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Pyramidology

University

Special Functions: Gamma Function:

$\Gamma(x)$ Can be thought of as a generalization to $x!$ when x is an integer.

(1) $\Gamma(x) = (x-1)!$; (2) x in the general case.

(2) Euler Defn.: $\Gamma(x) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots (n-1)}{x(x+1)(x+2)\cdots(x+n-1)} \cdot n^x$ *

from which we can think of $\Gamma(x)$ as

$$\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!}{x(x+1)(x+2)\cdots(x+n-1)} \cdot n^x = (n-1)!\, n^x \Big/ \lim_{n\to\infty} \prod_{k=0}^{n-1}(x+k)$$

$$\Gamma(x) = \lim_{n\to\infty} (n-1)!\, n^x \Big/ \prod_{n=0}^{n-1}(x+n)$$

$\Gamma(1) = \frac{n!}{n!} = 1$, $\Gamma(0) = \infty$, $\Gamma(-1) = \infty$

$\Gamma(-2) = \infty$, $\Gamma(-n) = \infty$

(4) Difference Formulae: $\Gamma(x+1) = x\,\Gamma(x)$ difference is (1)

$$\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{x(x+1)(x+2)\cdots(x+n-1)}$$

$$\Gamma(x+1) = \lim_{n\to\infty} \frac{(n-1)!\, n^{x+1}}{(x+1)(x+2)(x+3)\cdots(x+n)}.$$

$$\Gamma(x+1)/\Gamma(x) = \lim_{n\to\infty} \frac{n^x\, n}{n+x} = x \lim_{n\to\infty} \frac{n}{n+x} = x(1)$$

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \Gamma(1/2) = \sqrt{\pi}$$

$\Gamma(6) = \Gamma(5+1) = 5\,\Gamma(5)$; $\Gamma(11/2) = \Gamma(\frac{9}{2}+1) = \frac{9}{2}\Gamma(9/2)$

$= \frac{9}{2}\cdot\frac{7}{2}\Gamma(7/2) = \cdots = \frac{9}{2}\cdot\frac{1}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

* If m is a (+ve) integer $\Gamma(m+1) = m!$

By difference formula $\Gamma(m+1) = m\,\Gamma(m) = m(m-1)\,\Gamma(m-1) = m(m-1)(m-2)\cdots\Gamma(m-2) = m(m-1)(m-2)\cdots 2\times 1\,\Gamma(1) = m!\times 1$

(3) * Weierstrass Defn: $\Gamma(x)$ is defined by

$$1/\Gamma(x) = x\, e^{\gamma x} \prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right) e^{-x/n}\right\} \qquad (1)$$

$1/\Gamma(x) = x e^{\gamma x}\left(\prod_{n=1}\left(1+\frac{x}{n}\right) e^{-x/n}\right)$.

where $\gamma = \lim_{n\to\infty}\left(\sum \frac{1}{n} - \ln n\right) \approx 0.5772$

(Euler Const. γ)

The two definitions are equivalent, hence Prove.

* Connexion between the Gamma-function and the Circular functions.

(5) $\Gamma(x)\,\Gamma(1-x) = \pi / \sin \pi x$ $\quad\prod_{n=1}^{\infty}(1-(x/n)^2) = \sin(\pi x)/\pi x$

By Def. $\Gamma(x) = x^{-1} e^{-\gamma x} \prod_{n=1}^{\infty}\{(1+x/n)^{-1} e^{x/n}\}$

and $\Gamma(-x) = -x^{-1} e^{\gamma x} \prod_{n=1}^{\infty}\{(1-x/n)^{-1} e^{-x/n}\}$ $\Rightarrow \Gamma(x)\cdot\Gamma(-x) =$

$= -1/x^2 \prod_{n=1}^{\infty}\{(1-x^2/n^2)^{-1}\}$ $\Rightarrow \Gamma(x)\Gamma(-x) = -x^{-1}\prod_{n=1}^{\infty}(1-x^2/n^2)^{-1}$

$= \Gamma(x)\,\Gamma(1-x)$. but since $\prod_{n=1}^{\infty}(1-x^2/n^2) = \sin \pi x / \pi x \Rightarrow$

$\Gamma(x)\,\Gamma(1-x) = \pi/\sin \pi x \cdot \left(\Gamma(1/2)\right)^2 = \pi/\sin(\pi/2) = \pi$

(they add to one) $\Rightarrow \Gamma(1/2) = \sqrt{\pi}$.

Prove $\Gamma(1/3)\,\Gamma(1/3)\,\Gamma(2/3) = \pi/\sin(\pi/3)$

* Duplication Formulae. (Legendre) $\quad 2^{2x-1}\,\Gamma(x)\cdot\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$

(6) $2^{2x-1}\,\Gamma(x)\cdot\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$ i.e $2^{2x}\,\Gamma(x)\cdot\Gamma(x+1/2)/\Gamma(2x) = 2\sqrt{\pi}$.

By Euler Def. $\Gamma(x) = \lim_{n\to\infty} (n-1)!\, n^x / x(x+1)(x+2)\cdots(x+n-1)$

and $\Gamma(2x) = \lim_{n\to\infty} (2n-1)!\,(2n)^{2x} / 2x(2x+1)(2x+2)\cdots(2x+n-1)$

$\Gamma(x+1/2) = \lim_{n\to\infty} (n-1)!\, n^{x+1/2} / (x+1/2)(x+3/2)\cdots(x+\frac{n}{2}-1)$

Let n as $m \Rightarrow$ if $m = 2n$ $\quad \dfrac{2^{2x}\,\Gamma(x)\,\Gamma(x+\frac{1}{2})}{\Gamma(2x)} = 2\sqrt{\pi}$

Let $\phi(x) = 2^{2x}\cdot\Gamma(x)\cdot\Gamma(x+1/2)/\Gamma(2x)$

* Prove that the seq. $\{(\sum_{n=1}^{m} 1/n) - \ln m\}_{n=1}^{\infty}$ is a Convergent seq.

+ 1st To Prove the seq. is monotone $\Rightarrow u_{m+1} - u_m = \sum_{n=1}^{m+1}(1/n) - \ln(m+1) + \ln m$

$= 1/(m+1) - \ln(m+1) + \ln m = 1/(m+1) - (\ln(m+1) - \ln m) = 1/(m+1) - \ln\frac{m+1}{m} \Rightarrow$

$u_{m+1} - u_m = 1/(m+1) - \ln(\frac{1}{m}+1)$; but since $\left[\frac{x}{1+x} < \ln(x+1) < x\right] \Rightarrow$

$\frac{1}{m+1} = \frac{1/m}{1+1/m} < \ln(\frac{1}{m}+1) < 1/m \Rightarrow -\frac{1}{m+1} > -\ln(\frac{1}{m}+1) > -\frac{1}{m} \Rightarrow$

$0 = \frac{1}{m+1} - \frac{1}{m+1} > \frac{1}{m+1} - \ln(\frac{1}{m}+1) > \frac{1}{m+1} - \frac{1}{m} = \frac{m-m-1}{(m+1)m} = \frac{-1}{m(m+1)}$

Hence $\quad 0 > \frac{1}{m+1} - \ln(1/m+1) > \frac{-1}{m(m+1)} \Rightarrow$

$0 > u_{m+1} - u_m > -\frac{1}{m(m+1)}$, i.e $u_m > u_{m+1}$ ①

Therefore, the seq. is monotone non-increasing.

+ 2nd To Prove the seq. is bounded $\Rightarrow \sum_{n=1}^{m} 1/n - \ln m = \sum_{n=1}^{m} 1/n - \sum_{n=1}^{m-1}\int_n^{n+1}\frac{dt}{t}$

but since, $\frac{1}{n+1} < \ln\frac{n+1}{n} = \ln(1/n+1) < 1/n \Rightarrow -\frac{1}{n+1} > -\ln(1/n+1) > -\frac{1}{n}$

$\Rightarrow -\sum_{n=1}^{m-1}\frac{1}{n+1} > -\sum_{n=1}^{m-1}\int_n^{n+1}\frac{dt}{t} > -\sum_{n=1}^{m-1}\frac{1}{n} \Rightarrow$

$1 = \sum_{n=1}^{m} 1/n - \sum_{n=1}^{m-1}\frac{1}{n+1} > \sum_{n=1}^{m} 1/n - \ln(1/m+1) > \sum_{n=1}^{m} 1/n - \sum_{n=1}^{m-1} 1/n = 1/m$

Which means that ② $1 > u_m > 1/m > 0$, hence the seq. has an l.u.b = 1

, Sup. $u_n = 1$ and is bounded below by 0. Hence, $\{u_n\}_{n=1}^{\infty}$ is a Convergent sequence.

cos 3x = cos (2x + x)

Prove: $\sqrt{3}\,\Gamma(1/3)\,\Gamma(2/3) = \pi / \sin(\pi/3) = \sqrt{3}\,\Gamma(1/3)\,\Gamma(1-1/3) = \Gamma(1/3)\cdot(\pi/\sin(\pi/3))$

$$\Phi(x) = \frac{m^{mx}\left[\prod_{r=0}^{m-1}\Gamma\left(x+\frac{r}{m}\right)\right]}{\Gamma(mx)} = \sqrt{m}\,\sqrt{(2\pi)^{m-1}}$$

If p is an integer $\Rightarrow$

$$\Gamma(px) = \lim_{pn\to\infty} \frac{(pn-1)!}{\prod_{m=0}^{pn-1}(px+m)}\,(pn)^{px}$$

; * $\Phi(x) = \frac{m^{mx}\left[\prod_{r=0}^{m-1}\Gamma\left(x+\frac{r}{m}\right)\right]}{\Gamma(mx)} = \sqrt{m}\cdot\sqrt{(2\pi)^{m-1}}$

Prove: $e^{7/10}\sqrt{\pi}\,\Gamma(1/10) = \sqrt{5+\sqrt{5}}\;\Gamma(1/5)\cdot\Gamma(2/5)$

1. $2^{2x-1}\,\Gamma(x)\,\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$ [(m=2) above]

Let $x = 1/10$ $\Rightarrow$ $2^{-4/5}\,\Gamma(1/10)\,\Gamma(3/5) = \sqrt{\pi}\,\Gamma(1/5)$

$2^{-4/5}\,\Gamma(1/10)\times\left[\Gamma(2/5)\times\Gamma(3/5)\right] = \sqrt{\pi}\,\Gamma(1/5)\cdot\Gamma(2/5)$

$2^{-4/5}\,\Gamma(1/10)\times\frac{\pi}{\sin\frac{2\pi}{5}} = \sqrt{\pi}\,\Gamma(1/5)\,\Gamma(2/5)$

$\sin\theta = \cos(90-\theta)$

$2^{-4/5}\sqrt{\pi}\,\Gamma(1/10) = (\sin 72^\circ)\,\Gamma(1/5)\,\Gamma(2/5)$

2. $\sin 36^\circ = \cos 54^\circ = \cos(90^\circ - 36^\circ)$.

$\sin 2(18^\circ) = \cos(3\times 18^\circ)$

$\therefore 2\sin 18 = 4\cos^2 18 - 3 = 4 - 4\sin^2 18 - 3$

$4\sin^2 18 + 2\sin 18 - 1 = 0$

$\sin 18 = \frac{-2+\sqrt{4+16}}{8} = \frac{-1+\sqrt{5}}{4}$

$4\cos 18\cdot\cos^2 18 - 3\cos 18$

$\cos 36 = 4\cos^3\theta - 3\cos\theta$

3. $\sin 72^\circ = \cos 18 = \sqrt{1-\sin^2 18} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} =$

$\sin(90-18) =$

$\sqrt{\frac{10+2\sqrt{5}}{16}} = \frac{\sqrt{2}}{4}\sqrt{5+\sqrt{5}}$

$\therefore 2^{-4/5}\sqrt{\pi}\,\Gamma(1/10) = \frac{\sqrt{2}}{4}\sqrt{5+\sqrt{5}}\;\Gamma(1/5)\,\Gamma(2/5)$

$2^{2}\cdot 2^{-1/2}\cdot 2^{-4/5}\sqrt{\pi}\,\Gamma(1/10) = \sqrt{5+\sqrt{5}}\;\Gamma(1/5)\,\Gamma(2/5)$

$2^{4/2}\cdot 2^{-1/2}\cdot 2^{-3/2} = 2^{15/10}\cdot 2^{-8/10} = (+)\,2^{7/10}$

Since $\Gamma(x) = \lim_{n\to\infty}(n-1)!\,n^x / \prod_{k=0}^{n-1}(x+k)$, thus,

If p is an integer $\Rightarrow$ $\Gamma(px) = \lim_{pn\to\infty}(pn)^{px}[(pn-1)!/\prod_{k=0}^{pn-1}(px+k)$.

means that, $\Gamma(px) = \lim_{m\to\infty}(m-1)!\,m^{px}/\prod_{k=0}^{m-1}(px+k)$

i.e. m Plays the role of $n \Rightarrow$ if m is put as pn above,

we obtain. $m = pn \Rightarrow m-1 = pn-1$.

(4)

If p is an integer $\Rightarrow \Gamma(px) = \alpha (pn-1)!\,(pn)^{px} / \prod_{n=0}^{p} (px+np)$

$$\phi(x) = \frac{2^{2x}\,\Gamma(x)\,\Gamma(x+\frac{1}{2})}{\Gamma(2x)}$$ which is Constant

i.e. independent of x (i.e. x disappear)

Chosen for simplicity $\Rightarrow$ leads us to Known values

$$\phi(x) = \phi(\tfrac{1}{2}) = \frac{2^{2\cdot\frac{1}{2}}\,\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2})}{\Gamma(2\cdot\tfrac{1}{2})} = 2\sqrt{\pi}\cdot 1 / 1 = 2\sqrt{\pi}$$

If p is an integer, $\Gamma(px) = \lim_{pn\to\infty} (pn-1)!\,(pn)^{px} / \prod_{k=0}^{pn-1}(px+k)$ (2)

hence, $2^{2x-1}\,\Gamma(x)\,\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$

(Laguerre)

Solve $3^{3x}\,\Gamma(x)\,\Gamma(x+1/3)\,\Gamma(x+\tfrac{2}{3}) = 2\sqrt{3}\,\pi\,\Gamma(3x)$

By Euler's Def. $\Gamma(x) = \lim_{n\to\infty} (n-1)!\cdot n^x / x(x+1)(x+2)\cdots(x+n-1)$

$$\Gamma(x+1/3) = \lim_{n\to\infty} \frac{(n-1)!}{(x+1/3)(x+4/3)\cdots(x+1/3+n-1)} \times n^{x+1/3}$$

$$\Gamma(x+\tfrac{2}{3}) = \lim_{n\to\infty} (n-1)!\, n^{x+2/3} / (x+\tfrac{2}{3})(3+\tfrac{5}{3})\cdots(x+\tfrac{2}{3}+n-1)$$

$$\Gamma(3x) = \lim_{n\to\infty} \frac{(3n-1)!}{3x(3x+1)(3x+2)\cdots(3x+3n-1)}\,(3n)^{3x}$$

$\Rightarrow \phi(x) = 3^{3x}\,\Gamma(x)\,\Gamma(x+1/3)\,\Gamma(x+2/3) / \Gamma(3x) = 2\sqrt{3}\,\pi$

$\Rightarrow 3^{3x}\,\Gamma(x)\,\Gamma(x+1/3)\,\Gamma(x+2/3) = 2\sqrt{3}\,\pi\,\Gamma(3x)$

▶ $\phi(x) = 2\lim_{n\to\infty} x_n = 2$ (Constant independent of x) ◀

$\Rightarrow \phi(x) = \phi(1/3)$

$3\times\Gamma(1/3)\,\Gamma(2/3)\,\Gamma(1) = 3\pi / \sin(\pi/3) = 6\pi/\sqrt{3}$

hence in 1. $\Rightarrow \phi(x) = 2\sqrt{3}\,\pi$

* Definite Integral for $\Gamma(x)$. (7) $\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt$.

$$\int_0^\infty t^{x-1}\cdot e^{-t}\,dt = \Gamma(x) \quad ; x>0$$

« Eulerian Integrals »

evaluate (the Euler integral)

$I = \int_0^\infty e^{-x^2}\,dx$ $\Rightarrow$ let $t = x^2$ $\Rightarrow$ $dt = 2x\,dx$ $\Rightarrow$ $x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}}\,dt$

$I = \int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty t^{-1/2}e^{-t}\,dt = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$ $\quad x-1 = -\tfrac{1}{2} \Rightarrow x = \tfrac{1}{2}$

$\Rightarrow \int_0^\infty t^{x-1}e^{-t}\,dt = \Gamma(x)$

+ evaluate also $\int_0^\infty e^{-x^3}\,dx$ let $x^3 = t \Rightarrow$

$dt = 3x^2\,dx \Rightarrow dx = \frac{dt}{3x^2} = \frac{dt}{3t^{2/3}}$ $\quad -\tfrac{2}{3} = x-1 \Rightarrow \tfrac{3}{3}-\tfrac{2}{3} = x = \tfrac{1}{3}$

$\Rightarrow \frac{1}{3}\int_0^\infty e^{-t}\cdot t^{-2/3}\,dt = \Gamma(\tfrac{1}{3})$

$\Gamma(1/m) = \int_0^\infty e^{-x^m}\,dx = \int_0^\infty e^{-t}\cdot t^{\frac{m-1}{m}}\,dt = \int_0^\infty e^{-t}\cdot t^{1/m-1}\,dt = \int e^{-t}\cdot t^{1/m-1}\,dt$

$x^m = t \Rightarrow x = t^{1/m} \Rightarrow dt = m x^{m-1}\,dx \Rightarrow dx = \frac{1}{m}x^{-m+1}\,dt = \frac{1}{m}t^{\frac{-m+1}{m}}\,dt = \frac{1}{m}t^{1/m-1}\,dt$

Calculate $\int_0^\infty x e^{-x^2}\,dx$ (using two different methods)

let $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2}t^{-1/2}\,dt \Rightarrow$

$\int_0^\infty t^{-1/2}e^{-t}\times\frac{1}{2}t^{-1/2}\,dt = \frac{1}{2}\int_0^\infty t^{0}e^{-t}\,dt = \frac{1}{2}\Gamma(1) = \frac{1}{2}$ $\quad 1 = x-1,\ x = 2$

without using the Γ function. $(\int u\,dv = uv - \int v\,du)$.

$-\frac{1}{2}\int_0^\infty -2x e^{-x^2}\,dx \Rightarrow$ let $-2x e^{-x^2}\,dx = dv = de^{-x^2} \Rightarrow -\frac{1}{2}\int de^{-x^2} = -\frac{1}{2}[e^{-x^2}]_0^\infty = \frac{1}{2}$

$*\ -\frac{1}{2}\int x\times -2x e^{-x^2}\,dx = -\frac{1}{2}\int x\,de^{-x^2} = -\frac{1}{2}[x e^{-x^2} - \int e^{-x^2}\,dx]_0^\infty = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{4}\sqrt{\pi}$ ①

* let $x^m = t \Rightarrow x = t^{1/m} \Rightarrow dx = \frac{1}{m}t^{1/m-1}\,dt \Rightarrow \frac{1}{m}\int_0^\infty t^{n/m}\cdot t^{1/m-1}e^{-t}\,dt = \frac{1}{m}\int_0^\infty e^{-t}t^{\frac{n+1-m}{m}}\,dt = \Gamma(\frac{n+1}{m})$ ②

* evaluate $\int_0^\infty x^n e^{-x^m}\,dx$ ② ; $\int_0^\infty x^2 e^{-x^2}\,dx$ ①

notice that

$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx^2 = \frac{1}{2}\int_0^\infty e^{-u}\,du = \frac{1}{2}$

The advantage of using the Γ function in evaluating such integration as * ; for these integrands can not be done by elementary substitutions $\Rightarrow$ we use Γ (function) which represents the general case.

what made it a good substitution 1st the sign 2nd the root and how it is eliminated

(6)

Some techniques for evaluating integrals :-

1. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} \right]$

using either of the substitutions

$x = a \sin u$ or $x = a \tanh u$

=> let $x = a \tanh u$ =>

$a^2 + x^2 = a^2(1 - \tanh^2 u) = a^2 \operatorname{sech}^2 u$

$dx = a \operatorname{sech}^2 u\, du$ =>

$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \operatorname{sech}^3 u\, du$ => (by parts solved)

$\Rightarrow a^2 \int \operatorname{sech} u \cdot d\tanh u = a^2\left[\operatorname{sech} u \tanh u + \int \tanh^2 u \operatorname{sech} u \right]$ =>

($\tanh^2 u \to 1 - \operatorname{sech}^2 u$ of course this by guessing)

$\int \operatorname{sech}^3 u\, du = \operatorname{sech} u \tanh u + \int \operatorname{sech} u\, du - \int \operatorname{sech}^3 u\, du$ =>

2 $\int \operatorname{sech}^3 u\, du = \operatorname{sech} u \tanh u + \int \operatorname{sech} u\, du =$

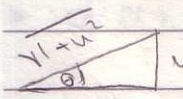
$\operatorname{sech} u \tanh u + \tan^{-1}(\sinh u)$ =>

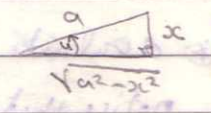
$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\left[\operatorname{sech} u \tanh u + \tan^{-1}(\sinh u) \right]$

since $\tanh u = x/a$ => $\operatorname{sech} u = \sqrt{1 - \tanh^2 u} = \frac{\sqrt{a^2 - x^2}}{a}$

$\sinh u = \frac{e^x - e^{-x}}{2}$

$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\left[\frac{x\sqrt{a^2 - x^2}}{a^2} + \tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}} \right) \right]$





$\tan^{-1} u = \sin^{-1} u / \sqrt{1 - u^2}$

$u = x/\sqrt{a^2 - x^2}$ => $\sqrt{1 + u^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{\sqrt{a^2 - x^2 + x^2}}{\sqrt{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}$

$u/\sqrt{1+u^2} = x/a$ *** [What are the conditions for the existance of $T^{-1} \equiv S$?]

(these conditions are dictated by T) *** $T(y) = R$, T is a differential operator

$T T^{-1} = T^{-1} T = I$ => $[T^{-1}(T(y)) = T^{-1}(R) = I(y) = y]$

In this case $T = (D^2 + PD + q)$, $D = \frac{d}{dx}$,

Thus the inverse of $(D^2 + PD + q) = T$ is $\left(\frac{1}{D^2 + PD + q}\right) = T^{-1} = S$

This is quite plausible after all T transforms a fn y into the other fn R

* Such a kind of functions can be solved by Circular or

an integral which appears frequently $I_n = \int_0^{\pi/2} \sin^n\theta \, d\theta$ n integer

the case when $n \neq$ integer $\Rightarrow$ β fun.

$I_n = \int_0^{\pi/2} \sin^{n-1}\theta \cdot \sin\theta \, d\theta = -\int_0^{\pi/2} \sin^{n-1}\theta \, d(\cos\theta) = -\left[\sin^{n-1}\theta \cos\theta\right]_0^{\pi/2} - \int \cos\theta \, d(\sin^{n-1}\theta)$

$= (n-1)\int_0^{\pi/2} \sin^{n-2}\theta \cos^2\theta \, d\theta = (n-1)\int_0^{\pi/2} \sin^{n-2}\theta (1-\sin^2\theta) \, d\theta \Rightarrow I_n = (n-1)\left[I_{n-2} - I_n\right]$

hence, $n I_n = (n-1) I_{n-2} \Rightarrow$

$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4}$ $\times$ $\frac{\pi}{2}$ even, 1 odd.

Linear Differential Equations of 2nd Order.

2nd degree polynomial has 2 roots $\Rightarrow$

* Can a 2nd order DE has 2 independent solutions?

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \quad (1)$$

let $D = \frac{d}{dx} \Rightarrow$

$$D^2 y + P(x) D + Q(x)\, y = R(x) \quad (2)$$

when $R(x) = 0 \Rightarrow$ Condition for u & v to be linearly independent $\left(\int_{-\infty}^{v} u\,v\,dx = 0\right)$:

$$D^2 y + P(x) D y + Q(x)\, y = 0 \quad (3)$$

the solution for this DE, (3), is called the Complementary f^{n}, and is written as $y_{C.F}$; this solution depends on two arbitrary Constants for the order of the DE is the 2nd. (Why)?

+ Suppose $y = u$, $y = v$ are two solutions for (3), therefore the general solution for (3) becomes $y_{c.f} = Au + Bv$ (2nd order dependent)

where A & B are arbitrary Constants.

+ Explanation:- Since $y = u$, $y = v$ are solutions for (3) $\Rightarrow$

$(D^2 + PD + Q)\,u = 0$ (4) and $(D^2 + PD + Q)\,v = 0$ (5)

Let $y = Au + Bv$, then

$(D^2 + PD + Q)(Au + Bv) = AD^2u + BD^2v +$

$PADu + PBDv +$

$QAu + QBv = 0$, for

$AD^2u + APDu + AQu = A \times 0 = 0$; also

$BD^2u + BPDu + BQv = B \times 0 = 0$.

hyperbolic functions, depending on which is doesn't give a lengthy solution.

Linear DE's with Constant Coefficients.

Let $P = 2a$, $Q = b$ $\Rightarrow$ (1) becomes

$(D^2 + 2aD + b)y = 0$ $\Rightarrow$ Let $y = e^{\lambda x}$ $\Rightarrow$

$(\lambda^2 + 2a\lambda + b)e^{\lambda x} = 0$, therefore $\lambda^2 + 2a\lambda + b = 0$

which is called the auxiliary equation, which can be found by replacing D by λ in the original DE, thus to solve

$y'' - 4y = 0 \Rightarrow (D^2 - 4)y = 0 \Rightarrow$

A.E. $(\lambda^2 - 4) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = +2$

$\Rightarrow y_{C.F} = A e^{-2x} + B e^{2x}$.

+ Solve:- $(D^2 - 4D - 5)y = 0 \Rightarrow$ A.E is $\lambda^2 - 4\lambda - 5 = 0 \Rightarrow$

$(\lambda + 1)(\lambda - 5) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 5 \Rightarrow$

$y_{C.F} = A e^{-x} + B e^{5x}$, where A, B are arbitrary Const's.

→ 1st Case: $\lambda_1 = \lambda_2 \Rightarrow \lambda^2 + 2a\lambda + b = 0 \Rightarrow$

$4a^2 - 4b = 0 \Rightarrow b = a^2 \Rightarrow (\lambda + a)^2 = \lambda^2 + 2a\lambda + a^2$

$\Rightarrow \lambda_1 = \lambda_2 = -a$; Hence the general solution is

$y = A e^{-ax} + B e^{-ax} = (A + B) e^{-ax} = C e^{-ax}$ and the solution depends on an arbitrary Const. C, therefore no longer a general solution $\Rightarrow$ Suppose the solution which we are looking for is

$y = u e^{-ax}$ (9) where u is the required f^n we are looking for

$\Rightarrow D^2(u e^{-ax}) + 2aD(u e^{-ax}) + a^2(u e^{-ax}) = 0$

$Dy = D(u e^{-ax}) = u' e^{-ax} + u(-a e^{-ax}) = (u' - au) e^{-ax}$ (10)

$D^2 y = [(u'' - au') e^{-ax} + (u' - au)(-a e^{-ax})] =$

$(u'' - 2au' + a^2 u) e^{-ax} = D^2 y \Rightarrow$ using (9) & (10) & (11) in

$(D^2 + 2aD + a^2) y = 0 \Rightarrow$

$e^{-ax}[(u'' - 2au' + a^2 u) + 2a(u' - au) + a^2 u] = 0$

$\Rightarrow [u''] = 0 \Rightarrow u' = A \Rightarrow u = Ax + B$

Therefore, the general solution becomes $y = (Ax + B) e^{-ax}$

where $Ax + B$ is a polynomial of 1st degree.

notice the DE, here, is of 2nd order $\Rightarrow$

A Generalization: we can consider the n order DE

$$a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n y = 0 \quad (12)$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = 0$$

we suppose $\lambda = \lambda_1$ which is a root that is repeated K times, i.e. the root $\lambda = \lambda_1$ is of multiplicity (K)

thus the auxiliary equation takes the form

$$(a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n) y = 0$$

this repeated root corresponds to the general solution for y which is $(A_0 x^{K-1} + A_1 x^{K-2} + \cdots A_K x^{K-1}) e^{\lambda_1 x}$

which is a polynomial of degree $(K-1)$ = degree of repitition $(K) - 1$.

+ Solve :-

$(D^2 + 4D + 4) y = 0 \Rightarrow$ A.E is $\lambda^2 + 4\lambda + 4 = 0$

the discriminent is $(\lambda + 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$

$\Rightarrow y = (Ax + B) e^{-2x}$.

+ Solve $(D^2 - 9) y = 0$, $D(D-1)^2 y = 0$

$(D^2 - 5D - 6) y = 0$, $(D+2)^5 y = 0$

Also Solve $2^{7/10} \sqrt{\pi}\, \Gamma(7/10) = \sqrt{5+\sqrt{5}}\, \Gamma(1/5) \cdot \Gamma(2/5)$.

notice, if $y = u e^{-ax} \Rightarrow Dy = D(u e^{-ax}) = u(-a e^{-ax}) + e^{-ax} u'$

$\Rightarrow D(u e^{-ax}) = (-au + u') e^{-ax}$

$D^2 y = D(-au + u') e^{-ax} = (u' - au) x - a e^{-ax} + e^{-ax}(u'' - au')$

$\Rightarrow D^2 y = (u'' - 2au' + a^2 u) e^{-ax}$

Hence $(D^2 + 2aD + a^2) y = 0$ becomes

$$e^{-ax}(u'' - 2au' + a^2 u + 2au' - 2a^2 u + a^2 u) = 0$$

that is to say $u'' = 0 \Rightarrow$

$$u' = A \quad \text{i.e.} \quad u = Ax + B .$$

tentative methods $\to a_1 x + a_2 x^2 + a_3 x + B_1 + \cdots$

* $\lambda(\lambda+3) e^{\lambda x} = 0 \Rightarrow \lambda = -3, 0 \Rightarrow y_{G.S} = A e^{-3x} + B e^{0x} = y_{C.F}$
* $\lambda(\lambda-1)^2 e^{\lambda x} = 0 \Rightarrow \lambda = 0, \lambda = 1, 1 \Rightarrow y_{G.S} = A + (A_1 x + B) e^{x} = y_{C.F}$
* $(\lambda - 6)(\lambda + 1) e^{\lambda x} \Rightarrow \lambda = 6, \lambda = -1 \Rightarrow y_{G.S} = A e^{6x} + B e^{-x} = y_{C.F}$
* $(\lambda + 2)^5 e^{\lambda x} = 0 \Rightarrow \lambda = -2, -2, -2, -2, -2 \Rightarrow (A x^4 + B x^3 + C x^2 + D x + E) e^{-2x}$

Solve $(D^2 + D - 6) y = 0$, $(D^2 - 9) y = 0$, $(D^3 - 7D^2 + 14D) y = 0$

y' D

(b)

Case 2:

There Can be an imaginary root for the auxiliary eqⁿ.

$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha_2 + i\beta$ its conjugate must appear.

For these two roots they correspond to a general solution as

$y = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}$ since $(e^{i\theta} = \cos\theta + i\sin\theta)$

$(e^{-i\theta} = \cos\theta - i\sin\theta)$

$\Rightarrow y = A_1 e^{(\alpha+i\beta)x} + A_2 e^{(\alpha-i\beta)x} =$

$e^{\alpha x}\left[A_1 e^{i\beta x} + A_2 e^{-i\beta x}\right] =$

$e^{\alpha x}\left[(A_1 + A_2)\cos(\beta x) + i(A_1 - A_2)\sin\beta x\right]$ = $A = A_1 + A_2$, $B = i(A_1 - A_2)$

$y = e^{\alpha x}\left[A\cos\beta x + B\sin(\beta x)\right]$

which Corresponds to the two conjugate imaginary roots $\alpha \pm i\beta$

in $e^{\alpha x}$ α represents the real part

and β represents the imaginary part.

example: Suppose that the roots of the auxiliary equation are

$5,\ 2+7i,\ 2-7i$

$y = Ae^{5x} + e^{2x}(B\cos 7x + C\sin 7x)$

$\frac{1}{2} \pm \frac{2}{3}i,\ 1 \pm \sqrt{2}i$

Solve: $y'' + 4y' + 13y = 0$

$\Rightarrow (D^2 + 4D + 13)y = 0$

$A.E = (\lambda^2 + 4\lambda + 13) = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4\times 13}}{2} = -2 \pm$

which Can be solved if $\sqrt{b^2 - 4ac}$ is a Complete square i.e. $\sqrt{\alpha} = \sqrt{\beta^2}$

$a = 1,\ b = 4,\ c = 13 \Rightarrow b^2 - 4ac = 16 - 52 = -36$

$\Rightarrow AE = [\lambda + (2 + 3i)]$

$\lambda_{1,2} = -2 + 3i,\ -2 - 3i$

$e^{-2x}(A\cos 3x + B\sin 3x)$

Solve $(D^2 + D + 1)y = 0$

[c] Bernoulli's method

Solve: $y^{(4)} - y'' - 2y' + 2y = 0$

$(D^4 - D^2 - 2D + 2)y = 0 \Rightarrow$ A.E $= \lambda^4 - \lambda^2 - 2\lambda + 2 = 0$

$\lambda^2(\lambda^2 - 1) - 2(\lambda - 1) = \lambda^2(\lambda-1)(\lambda+1) - 2(\lambda-1) = (\lambda-1)[\lambda^2(\lambda+1) - 2] = (\lambda-1)[\lambda^3 + \lambda^2 - 2] = 0$

$(\lambda-1)(a\lambda^2 + b\lambda + c) = a\lambda^3 + b\lambda^2 + c\lambda - a\lambda^2 - b\lambda - c = a\lambda^3 + (b-a)\lambda^2 + (c-b)\lambda - c =$

$\lambda^3 + \lambda^2 - 2 = 0 \Rightarrow -a = 1, b - a = 1 \Rightarrow b = 2$; $c - b = 0 \Rightarrow c = 2 \Rightarrow$

$\lambda^4 - \lambda^2 - 2\lambda + 2 = (\lambda-1)^2(\lambda^2 + 2\lambda + 2) = 0$; $(\lambda^2 + 2\lambda + 2) = 0 \Rightarrow \frac{-2 \pm \sqrt{4-8}}{2} = \lambda = -1 \pm i$

$\lambda = 1, 1, -1+i, -1-i \Rightarrow y_{G.S} = (Ax + B)e^x + e^{-x}(A_1 \cos x + B_1 \sin x)$

if the imaginary roots $\alpha \pm i\beta$ each one of the independetly (K) times.

repeated roots $\Rightarrow$

$e^{\alpha x}[A_0 + A_1 x + \dots A_{K-1} x^{K-1}]$

Always insert a guessed value of a root when it is hard to factorize directly. ($\lambda \in \{\pm i, \pm 1, \pm 2, \text{small values}\}$)

1. $y'' - 2y' - 4y = 0$
2. $y'' + 2y' + 5y = 0$
3. $y'' + 6y' + 18y = 0$
4. $(D^2 + 9)y = 0$
5. $(D^2 + \dots - 4)y = 0$
   $(D^3 - D^2 - 2D)y = 0$
6. $(D^4 - 8D^2 - 9)y = 0$
   or (-)
   $(D^4 + 10D^2 + 9)y = 0$

solve: $(D^2 + a^2)y = 0$

Rules of finding the special solutions:

Rules for finding the special solutions

1. $\phi(D)[e^{\alpha x}] = e^{\alpha x}\phi(\alpha)$

$\phi(D)\cdot\begin{Bmatrix}\sin\alpha x\\ \cos\alpha x\end{Bmatrix} = \begin{Bmatrix}\sin\alpha x \cdot \phi(-\alpha^2)\\ \cos\alpha x \cdot \phi(-\alpha^2)\end{Bmatrix}$

2. $\phi(D^2) \times \begin{Bmatrix}\sin\alpha x\\ \cos\alpha x\end{Bmatrix} = \begin{Bmatrix}\sin\alpha x \times \phi(-a^2)\\ \cos(ax) \times \phi(-a^2)\end{Bmatrix}$

$\phi(D)[e^{\alpha x} v] = e^{\alpha x}\phi(D+\alpha)v$

3. $\phi(D)[e^{\alpha x} \times v] = e^{\alpha x}\phi(D+\alpha)v$ (coefficient of x)

v-1 imp)

$D\phi(D)e^{\alpha x}v = e^{\alpha x}\phi(D)v + e^{\alpha x}\phi(\alpha)v + e^{\alpha x}\phi(D)v = e^{\alpha x}(\phi(D+\alpha)v)$

Solve $(D^2 - 6D + 5)y = e^{2x}$

1st we find the Complementary f.n.

suppose $(D^2 - 6D + 5)y = 0 \Rightarrow$ A.E $= \lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 5$

$y_{C.F} = Ae^x + Be^{5x}$.

$y_{G.S} = y_{C.F} + y_{P.I}$

2nd P.I Particular Integral ... i.e there is no arbitrary const*

$y_{P.I} = \left(\frac{1}{D^2 - 6D + 5}\right) e^{2x}$; $\phi(D) = 1/(D^2 - 6D + 5)$

just a notation = (no real division)

$\Rightarrow y_{P.I} = e^{2x}/(2^2 - 6*2 + 5) = -e^{2x}/3$

$\Rightarrow y_{G.S} = y_{C.F} + y_{P.I} = Ae^{x} + Be^{5x} - \frac{1}{3}e^{2x}$

Ex. $(D^2 + 5)\sin 3x = (-3^2 + 5)\sin 3x = -4\sin 3x$

which is according to the rule. But,

Or $(D^2 + 5)\sin 3x = D^2(\sin 3x) + 5\sin 3x =$

$D\sin 3x = 3\cos 3x \Rightarrow D^2 \sin 3x = -9\sin 3x$

Ex. $\frac{1}{D^2 + 4}\cos 2x = \frac{\cos 2x}{-2^2 + 4} = \frac{\cos 2x}{0}$

$\rightarrow \frac{1}{D^4 + 3D^2 + 8}\sin 5x = \frac{\sin 5x}{(-5)^2 + 3(-5^2) + 8}$

* $\rightarrow \frac{1}{D^3 + 1}\cos 2x = \frac{1}{D(D^2 + 1)} = \frac{1}{D(-2^2) + 1}\cos 2x =$

$\frac{1}{1 - 4D}\cos 2x$

$*(D^2 - 2D - 4)y = 0 \Rightarrow \lambda^2 - 2\lambda - 4 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4+16}}{2} = 1 \pm \sqrt{5} \Rightarrow y = Ae^{(1+\sqrt{5})x} + Be^{(1-\sqrt{5})x}$.

$*(D^2 + 2D + 5)y = 0 \Rightarrow \lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i \Rightarrow y = (A\cos 2x + B\sin 2x)e^{-x}$

$*(D^2 + 6D + 18)y = 0 \Rightarrow (\lambda^2 + 6\lambda + 18) = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36 - 4*18}}{2} = -3 \pm 3i \Rightarrow y = (A\cos 3x + B\sin 3x)e^{-3x}$

$*(D^2 + 9)y = 0 \Rightarrow D^2 + 9 = 0 \Rightarrow \lambda = \frac{\pm 6i}{2} \Rightarrow \lambda = \pm 3i \Rightarrow y = A\cos 3x + B\sin 3x$.

$*(D^3 - D^2 - 2D)y = 0 \Rightarrow \lambda(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda(\lambda - 2)(\lambda + 1) = 0 \Rightarrow y_{G.S} = A + Be^{2x} + Ce^{-x}$.

$*(D^4 - 8D^2 - 9)y = 0 \Rightarrow (\lambda^2)^2 - 8(\lambda^2) - 9 = 0 \Rightarrow (\lambda^2 - 9)(\lambda^2 + 1) = 0 \Rightarrow (\lambda + 3)(\lambda - 3)(\lambda + i)(\lambda - i) = 0$

$y_{G.S} = Ae^{-3x} + Be^{3x} + (A_1\cos x + B_1\sin x)$.

$*(D^4 + 10D^2 + 9)y = 0 \Rightarrow (\lambda^2)^2 + 10(\lambda^2) + 9 = 0 \Rightarrow (\lambda^2 + 9)(\lambda^2 + 1) = 0 \Rightarrow$

$(\lambda + 3i)(\lambda - 3i)(\lambda + i)(\lambda - i) = 0 \Rightarrow y_{G.S} = (A\cos 3x + B\sin 3x) + (A_1\cos x + B_1\sin x)$

Notice: ① $\cosh x = \frac{e^x + e^{-x}}{2}$, ② $\sinh x = \frac{e^x - e^{-x}}{2}$.

① + ② = $e^x = \cosh x + \sinh x$; ① − ② = $e^{-x} = \cosh x - \sinh x$

Can be used for solving ($D^2 y = a^2 y$)

$D = \frac{d}{dx} \Rightarrow \frac{1}{D} = \frac{dx}{d} = \int dx$ $\quad D^n \ln(ax+b) = (-)^{n-1}(n-1)!\, a^n / (ax+b)^n$

$\Rightarrow D^2(e^{3x}\sin x) = e^{3x}((D+3)^2 \sin x) = e^{3x}(D^2+6D+9)\sin x$

$= e^{3x}[-1^2 + 6D + 9]\sin x = e^{3x}(8+6D)\sin x$

$\Rightarrow (2D+1)^4\left[e^{-\frac{1}{2}x}\ln x\right] = e^{-\frac{1}{2}x}\left[2(D-\frac{1}{2})+1\right]^4 \ln x =$

$e^{-\frac{1}{2}x}[16\,D^4]\ln x$

Rule $\Rightarrow$ to get $D^n \ln(ax+b) = \dfrac{(-)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

$\Rightarrow 16\, e^{-\frac{1}{2}x} \times \dfrac{(-)^3\, 3!}{x^4} = -3!\,16\, e^{-x/2}/x^4$.

$\Rightarrow$ (V.Imp) $\dfrac{1}{D-1}\dfrac{e^{2x}}{1+e^x} = \dfrac{1}{D-1}\left[e^x \cdot \dfrac{e^x}{1+e^x}\right]$ (1)

$= e^x \dfrac{1}{D-1+1}\dfrac{e^x}{1+e^x} =$ (Notice the technique)

$e^x \dfrac{1}{D}\dfrac{e^x}{1+e^x} = e^x \int \dfrac{e^x\,dx}{1+e^x} = e^x \ln(1+e^x)$

which ends the problem as a DE

(+) which means if it necessitates that we want to convert the operator into an integration we do any thing to say e^x / e^{-x} (2)

$\Rightarrow \dfrac{1}{D^2-1}\cdot\dfrac{e^{2x}}{1+e^x}$ all conventional trials are complicating the problem

so we have to convert $\frac{1}{(D^2-1)} = \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\frac{e^{2x}}{1+e^x}$ (3)

i.e by Partial fractions

$= \frac{1}{2}[I_1 - I_2]$

Since $I_1 = \dfrac{1}{D-1}\dfrac{e^{2x}}{1+e^x} = e^x \ln(1+e^x)$ (From Previously)

$I_2 = \dfrac{1}{D+1}\dfrac{e^{2x}}{1+e^x} = \dfrac{e^{-x}}{D+1-1}\cdot\dfrac{e^{3x}}{1+e^x}$ (2)

$e^{-x}\int \dfrac{e^{3x}\,dx}{1+e^x}$ (did you notice the technique used to get rid of $(D+1)$ and convert it into integration)

(14)

Solve: $\frac{(D+1)^2}{(D-1)}(\cosh x) \Rightarrow = \frac{3}{2}\left[e^x + xe^{-x}\right] + \frac{1}{4}e^{-x}$

$I_2 = e^{-x}\int \frac{e^{3x}\,dx}{1+e^x}$; let $e^x = t \Rightarrow$

$e^x\,dx = dt \Rightarrow dx = dt/t \Rightarrow$

$\int \frac{t^2\,dt}{t+1} = \int \frac{(t^2-1)+1}{t+1}\,dt =$

$\int (t-1)\,dt + \int \frac{dt}{t+1} = \frac{1}{2}t^2 - t + \ln(1+t) =$

$\frac{1}{2}[I_1 - I_2] = \frac{1}{2}\left[e^x \ln(1+e^x) - \frac{e^{-x}}{2}(e^x+1)^2 - e^{-x}\ln(1+e^x)\right]$

* Solve $(D^3 + D^2 - D - 1)y = 2\cosh x$

1st Complementary fn. i.e using Auxiliary eqtn

A.E $= (\lambda^3 + \lambda^2 - \lambda - 1) = 0$

$\lambda^2(\lambda+1) + (\lambda+1) = (\lambda+1)(\lambda^2+1) =$

$(\lambda+1)(\lambda+1)(\lambda-1) = 0$

$\lambda = +1, -1, -1$ (repeated twice)(-1)

$y_{C.F} = (Ax+B)e^{-x} + Ce^x$ ①

but the particular Integral is still wanted.

$y_{P.I} = \frac{1}{D^3+D^2-D-1}\,2\cosh x =$

$\frac{1}{(D+1)^2(D-1)}\cdot(e^x + e^{-x}) = I_1 + I_2$

$I_1 = \frac{1}{(D+1)^2(D-1)}e^x = \frac{1}{D-1}\left[\frac{1}{(D+1)^2}e^x\right]$ (technique

$\frac{1}{D-1}\frac{e^x}{4} = e^x\left(\frac{1}{D+1-1}\frac{1}{4}\right) = e^x\int \frac{1}{4}dx = \frac{xe^x}{4}$.

$I_2 = \frac{1}{(D+1)^2(D-1)}e^{-x} = \frac{1}{(D+1)^2}\left[\frac{1}{D-1}e^{-x}\right] =$

$\frac{1}{(D+1)^2}\cdot\frac{e^{-x}}{-2} = -e^{-x}\frac{1}{D^2}\frac{1}{2} = e^{-x}\iint \frac{1}{2}(dx)^2 =$

$-\frac{e^{-x}}{2}\cdot\frac{x^2}{2} = -\frac{x^2e^{-x}}{4} \Rightarrow y_{P.I} = \frac{xe^x}{4} - \frac{x^2e^{-x}}{4}$

$y_{G.S} = y_{C.F} + y_{P.I} =$

Solve $\frac{1}{(D-1)(D+1)^2}\cosh x$

$$\Gamma(px) = \lim_{pn\to\infty} (pn-1)!\,(pn)^{px} \Big/ \prod_{k=0}^{pn-1}(px+k)$$

$$(Ax+B)e^x + Ce^x + \frac{1}{4}x(e^x - xe^x)$$

(6) Gauss Multiplication formulae « of Gamma fns »

$\Gamma(x)\,\Gamma(x+1/m)\,\Gamma(x+2/m) \cdots \Gamma(x+\frac{m-1}{m}) =$

$(2\pi)^{\frac{m-1}{2}}\; m^{\frac{1}{2}-mx}\,\Gamma(mx)$ (observation, sensation)

if we can find

$$\phi(x) = m^{mx}\,\frac{\Gamma(x)\,\Gamma(x+1/m)\cdots\Gamma(x+\frac{m-1}{m})}{\Gamma(mx)}$$

(6) $\phi(x) = m^{mx}\left[\prod_{r=0}^{m-1}\Gamma(x+r/m)\right]/\Gamma(mx) =$

$m^{1/2}/(2\pi)^{m-1}$

by Euler Def. « m is Constant »

« r is Const. w.r.t $\Gamma(x+\frac{r}{m})$ »

$$\Gamma(x+\tfrac{r}{m}) = \lim_{n\to\infty} \frac{(n-1)!\,n^x}{x(x+\frac{1}{m})\cdots(x+\frac{r}{m})\cdots(x+\frac{n-1}{m})} \times n^{r/m}$$

$$\prod_{r=0}^{m-1}\Gamma(x+\tfrac{r}{m}) = \left(x+\tfrac{r}{m}+k = (mx+r+km)/m\right)^{k=0}; \quad \prod_{r=0}^{m-1} n^{r/m} = n^{\left(0+\frac{1}{m}+\frac{2}{m}+\cdots+\frac{m-1}{m}\right)} = n^{\frac{1}{m}\left(\frac{m-1}{2}r\right)}$$

$$\prod_{r=0}^{m-1}\left\{\lim_{n\to\infty} \frac{(n-1)!\,n^{mx}}{(mx+r)(mx+r+m)\cdots(mx+r+mn-m)} \times n^{x+r/m}\right\} =$$

$$\lim_{n\to\infty} \frac{[(n-1)!]^m\; a(m^{mn}) \times (n^{mx})\; n^{\frac{1}{m}\cdot\frac{m}{2}(m-1)}}{mx(mx+1)\cdots(mx+m-1)\,|\,(mx+m)(mx+m+1)\cdots(mx+mn-1)} \quad (1)$$

arithmetic Progression

also $\Gamma(mx) = \lim_{N\to\infty} \frac{(N-1)!\times N^{mx}}{mx(mx+1)(mx+2)\cdots(mx+N-1)}$

taking $\lfloor N = mn \rfloor$ ⇒

$$\Gamma(mx) = \lim_{mn\to\infty} \frac{(mn-1)!\;(mn)^{mx}}{mx(mx+1)\cdots(mx+mn-1)} \quad (2)$$

$$\Rightarrow \phi(x) = \frac{m^{mx}\,(1)}{(2)} = m^{mx}\lim_{n\to\infty}\left[\frac{(n-1)!}{(mn-1)!}\right]^m \cdot \frac{m^{mn}\cdot n^{mx}\cdot n^{\left(\frac{m-1}{2}\right)}}{(mn)^{mx}}$$

which is correct if x disappears i.e $\phi(x)$ is independent of (x) ⇒ $\phi(x) = \lim_{n\to\infty}\left[\frac{(n-1)!}{(mn-1)!}\right]^m \cdot m^{mn}\cdot n^{\left(\frac{m-1}{2}\right)}$

$\phi(x) = $ Const. (independent of x). $\phi(x) = \phi(\frac{1}{m})$

$\sum_{r=0}^{m-1} r = \frac{m-1}{2}(1+m-1) = \frac{m(m-1)}{2}$

(16)

$$\prod_{r=1}^{m-1} \sin\left(\frac{\pi r}{m}\right) = \frac{m}{2^{m-1}}$$

$$\prod_{r=1}^{m-1}\left(\prod_{k=0}^{n-1}(r+mk)\right) = \text{(operate as such)}$$

$k=0,\ r:1\to m-1$

$k=1,\ r:1\to m-1$

$k=n-1,\ r:1\to m-1$

$m^{mx-x}\,2\,1/m$

$\Rightarrow \phi(1/m) = m\,\Gamma(1/m)\,\Gamma(2/m)\cdots = m^{m/m}\prod_{r=1}^{m-1}\Gamma\left(\frac{r}{m}\right)$

$$\prod_{r=1}^{m-1}\Gamma(r/m) = \prod_{r=1}^{m-1}\left[\frac{(m-1)!\,n^{r/m}}{\prod_{k=0}^{m-1}(\frac{r}{m}+k)}\right] = \left[(m-1)!\right]^{m}\cdot m^{\frac{m(m-1)}{2}}\cdot\frac{m^{m(m-1)}}{(mn-1)!/(n-1)!}$$

$$\phi(1/m) = m\,\Gamma(1/m)\,\Gamma(2/m)\cdots\Gamma(1-1/m)$$

$$\phi(1/m) = m\,\Gamma\left(1-\tfrac{1}{m}\right)\Gamma\left(1-\tfrac{2}{m}\right)\cdots\Gamma(1/m) = \frac{m^{m}\cdot m^{m-1}}{(m-1)!}$$

$$= \left[\text{using the law } \Gamma(x)\,\Gamma(1-x) = \pi/\sin(\pi x)\right]$$

$$\Rightarrow \left[\phi(1/m)\right]^2 = m^2\prod_{r=1}^{m-1}\frac{\pi}{\sin\frac{\pi r}{m}}$$

$$\left\{\frac{(1-\cdots-(m-1))(m+1-\cdots-(m+m-1))}{(2m+1-\cdots)}\right\}$$

$$\frac{\pi}{\sin\frac{\pi}{m}}\cdot\frac{\pi}{\sin\frac{2\pi}{m}}\cdot\frac{\pi}{\sin\frac{3\pi}{m}}\cdots\frac{\pi}{\sin\frac{(m-1)\pi}{m}}\times m^2$$

$(m-1)$ times

$$\phi(1/m) = \frac{\pi^{\frac{m-1}{2}}\sqrt{m}}{\left[\sin\frac{\pi}{m}\cdot\sin\frac{2\pi}{m}\cdots\sin\left(\frac{m-1}{m}\pi\right)\right]^{1/2}}$$

$$\sin\frac{\pi}{m}\cdot\sin\frac{2\pi}{m}\cdots\sin\frac{m-1}{m}\pi = \frac{m}{2^{m-1}}$$

$$\Rightarrow \phi(1/m) = \frac{m\,\pi^{(m-1)/2}\cdot 2^{(m-1)/2}}{m^{1/2}} = \sqrt{m\,\pi^{m-1}\,2^{m-1}} = \sqrt{m(2\pi)^{m-1}}$$

$$= m^{1/2}\sqrt{(2\pi)^{m-1}}$$

$$\Rightarrow \Gamma(x)\,\Gamma(x+1/m)\,\Gamma(x+2/m)\cdots\Gamma(x+\tfrac{m-1}{m}) = (2\pi)^{(m-1)/2}\cdot m^{(1/2-mx)}\,\Gamma(mx)$$

$$\text{denominator} = \prod_{s=0}^{mn-1}(mx+s)$$

$$\phi(x) = m^{mx}\left[\prod_{r=0}^{m-1}\Gamma\left(x+\frac{r}{m}\right)\right]/\Gamma(mx) = m^{1/2}\sqrt{(2\pi)^{m-1}}$$

$$\prod_{r=0}^{m-1}\left[\Gamma\left(x+\frac{r}{m}\right)\right] = \prod_{r=0}^{m-1}\left[\frac{(n-1)!\times n^{(x+\frac{r}{m})}}{\prod_{k=0}^{n-1}(x+\frac{r}{m}+k)}\right] = \prod_{r=0}^{m-1}\left[\frac{(n-1)!\cdot n^{x}\cdot n^{r/m}\cdot m^{n}}{\prod_{k=0}^{n-1}(mx+km+r)}\right] =$$

the numerator will be $\prod_{r=0}^{m-1}\left((n-1)!\cdot m^{n}\cdot n^{x}\cdot n^{r/m}\right)$

$\left[(n-1)!\right]^{m}\cdot m^{mn}\cdot n^{mx}\cdot n^{\frac{1}{m}(1+2+\cdots m-1)}$ $\Rightarrow$ since $1+2+3+\cdots(m-1) = \frac{m-1}{2}(m-1+1)$

numerator $= \left[(n-1)!\right]^{m}\cdot m^{mn}\cdot n^{mx}\cdot n^{\frac{m-1}{2}}$

denominator $= [mx(mx+1)(mx+2)\cdots(mx+m-1)][(mx+m)(mx+m+1)(mx+m+2)\cdots(mx+2m+m-1)]\cdots[(mx+(n-1)m)(mx+(n-1)m+1)\cdots(mx+(n-1)m+m-1)]$

($r:0\to m-1$, $k=0$) ($r:0\to m-1$, $k=1$) ($r:0\to m-1$, $k=n-1$)

$= mx+mn-m+m-1$

Beta function: $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$.

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad p>1,\ q>1$$

(1) Symmetry Property $\quad B(p,q) = B(q,p)$

$$B(q,p) = B(p,q) \quad \text{e.g.} \quad B(3,\tfrac{1}{4}) = B(\tfrac{1}{4},3)$$

Proof. Put $[t = 1-T]$, $dt = -dT$

$$B(p,q) = \int_1^0 (1-T)^{p-1}\,T^{q-1}(-dT) = \int_0^1 T^{q-1}(1-T)^{p-1}\,dT$$

$$= B(q,p)$$

(2) $\quad pB(p,q+1) = qB(p+1,q)$

$$pB(p,q+1) = qB(p+1,q)$$

By definition $\quad B(p,q+1) = \int_0^1 t^{p-1}(1-t)^{q+1}\,dt =$

$$\int_0^1 (1-t)^{q+1}\,d\left(\frac{t^p}{p}\right) = (1-t)^{q+1}\cdot\frac{t^p}{p}\Big|_0^1 + \frac{q+1}{p}\int_0^1 t^p(1-t)^{q}\,dt$$

$$B(p,q+1) = \frac{q+1}{p}\int_0^1 t^{p+1-1}(1-t)^{q-1}\,dt$$

$$\therefore\ B(p,q+1) = \frac{q}{p}B(p+1,q) \quad \text{hence}$$

$$pB(p,q+1) = qB(p+1,q).$$

(3) $\quad B(p,q+1) + B(p+1,q) = B(p,q)$

$$B(p,q+1) + B(p+1,q) = B(p,q)$$

$\int_0^{\pi/2} \sin^{7}\theta\cdot\cos^{5}\theta\,d\theta$ $\quad 2p-1=7 \Rightarrow p=4$, $\quad 2q-1=5 \Rightarrow q=3$ $\quad \Gamma(n+1) = n\,\Gamma(n)$

$\frac{1}{2}\Gamma(4)\cdot\Gamma(3)/\Gamma(7) = \ldots$, $\quad \Gamma(3) = \Gamma(2+1) = 2\Gamma(2)$, $\quad \Gamma(7) = \Gamma(6+1) = 6\Gamma(6)$

$3!\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}/\ldots = (2^3\cdot 3!/11\cdot 9\cdot 7\cdot 5) = \int_0^{\pi/2}\sin^7\theta\cdot\cos^5\theta\,d\theta$

$= \int_0^{\pi/2}\sin^{3/2}\theta\cdot\cos^{-1/2}\theta\,d\theta$, $\quad 2p-1 = 3/2 \Rightarrow p = \frac{5}{4}$, $\quad 2q-1 = -\frac{1}{2} \Rightarrow q = \frac{1}{4}$

$\frac{1}{2}\dfrac{\Gamma(5/4)\cdot\Gamma(1/4)}{\Gamma(1)}$ $\quad \Gamma(x)\cdot\Gamma(1-x) = \pi/\sin(\pi x) \Rightarrow$

$\frac{1}{2}(\pi/\sin(3\pi/4)) = \int_0^{\pi/2}\sqrt{\frac{\sin\theta}{\cos\theta}}\,dx$

(4) $\quad B(p,q) = \Gamma(p)\cdot\Gamma(q)/\Gamma(p+q)$

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} \qquad (\text{N.I.m } p)$$

$(1-\sin^2\theta = \cos^2\theta)$

(5) $$\int_0^{\pi/2}\sin^{2p-1}\theta\,\cos^{2q-1}\theta\,d\theta = \frac{1}{2}B(p,q) = \frac{1}{2}\cdot\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

This Eqtn enables us to find the value of integrals in case of the power of $\sin\theta$ or $\cos\theta$ is not an integer. $(\sin^2\theta = t \Rightarrow dt = 2\sin\theta\cdot\cos\theta\,d\theta)$

$$\int_0^{\pi/2}\sin^{2p-1}\theta\cdot\cos^{2q-1}\theta\,d\theta = \frac{1}{2}B(p,q) = \frac{1}{2}(\Gamma(p)\cdot\Gamma(q)/\Gamma(p+q)).$$

From the Def. of $B(p,q)$ $\quad B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$

Put $\boxed{t = \sin^2\theta}$ $\Rightarrow dt = 2\sin\theta\cos\theta\,d\theta$

$\Rightarrow B(p,q) = \int_0^{\pi/2} \sin^{2p-2}\theta\,\cos^{2q-2}\theta \cdot 2\sin\theta\cos\theta\,d\theta =$

$$2\int_0^{\pi/2} \sin^{2p-1}\theta\,\cos^{2q-1}\theta\,d\theta$$

e.g. $\int_0^{\pi/2} \sin^{3/2}\theta\cdot\cos^{5/4}\theta\,d\theta = \frac{1}{2}B\left(\frac{5}{4},\frac{9}{8}\right) = \frac{1}{2}\cdot\frac{\Gamma(5/4)\cdot\Gamma(9/8)}{\Gamma(5/4+9/8)}$

Solve $\int_0^{\pi/2} \sin^4\theta\,\cos^6\theta\,d\theta = \frac{3\cdot1\times5\cdot3\cdot1}{10\cdot8\times6\cdot4\cdot2}\times\frac{\pi}{2}$

or $\frac{1}{2}B\left(\frac{5}{2},\frac{7}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(\frac{5}{2}+\frac{7}{2})} = \frac{1}{2}\cdot\frac{3/2\cdot1/2\,\Gamma(1/2)\times 5/2\cdot3/2\cdot1/2\,\Gamma(1/2)}{5!} =$

$$\frac{3\cdot1\cdot5\cdot3\cdot1}{2^4\cdot2\times5\cdot4\cdot3\cdot2\cdot1}\cdot\frac{\pi}{2}$$

Solve: $\int_0^{\pi/2} \sin^7\theta\,\cos^3\theta\,d\theta$; $\int_0^{\pi/2}\sqrt{\frac{\sin\theta}{\cos\theta}}\,d\theta$

$2p-1=7 \Rightarrow 2p=8 \Rightarrow p=4$ $\quad 2q-1=3 \Rightarrow 2q=4 \Rightarrow q=2$

$\Rightarrow 2\cdot\frac{1}{2}\cdot\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} = \frac{3!\,1!}{5!} = \frac{1}{40}$

$2p-1=\frac{1}{2} \Rightarrow 2p=\frac{3}{2} \Rightarrow p=\frac{3}{4}$ $\quad 2q-1=-\frac{1}{2} \Rightarrow 2q=\frac{1}{2} \Rightarrow q=\frac{1}{4}$

$\Rightarrow \frac{1}{2}\cdot\frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(5/4)}$

Solve: $D(D^2-1)(2D+3)y = \sin x + 5$

Auxiliary eqtn $\quad D(D^2-1)(2D+3) = 0$

$D = -\frac{3}{2}, -1, 0, 1$

$$y_{C.F} = Ae^{-3x/2} + Be^{-x} + C + A'e^{x}.$$

$$y_{P.I} = \frac{1}{D(D^2-1)(2D+3)}(\sin x + 5)$$

I_1 $\quad I_2$

$$I_2 = \frac{1}{D(D^2-1)(2D+3)}\,5 =$$

$$\frac{1}{D}\left[\frac{5e^{0x}}{(0^2-1)(2\cdot0+3)}\right] = \frac{1}{D}\left[\frac{5e^{0x}}{(0-1)(0+3)}\right] = -\frac{5}{3}\cdot\frac{1}{D}(1) = -\frac{5}{3}x = I_1$$

$$I_2 = \frac{1}{D(D^2-1)(2D+3)}\sin x = \frac{1}{(D^2-1)(2D^2+3D)}\sin x$$

$\frac{1}{D(2D+3)}$ $\quad \frac{1}{D^2-1}$ $\quad \frac{1}{2}\cdot\alpha$ $\quad \frac{1}{D(2D+3)}\frac{\sin x}{-2}$ $\quad -\frac{1}{2}\frac{2D-3}{D(4D^2-9)}\frac{\sin x}{-(2-3)}$ $\quad \frac{\sin x}{+13\times2}$ $=$

Like multiplication by the conjugate $\Rightarrow$ $\quad \frac{1}{13}\sin x + \frac{3}{13\times2}\int\sin x\,dx$

$$-\frac{1}{2}\cdot\frac{3D+2}{9D^2-4}\sin x = -\frac{1}{2}\cdot\frac{3D+2}{-9-4}\sin x = \frac{1}{13}\sin x + \frac{3}{13\times2}\cos x = I_2$$

$$y_{G.S} = y_{C.F} + y_{P.I} = y_{C.F} + I_1 + I_2$$

If $f(-1)=0 \Rightarrow f'(\lambda)|_{\lambda=-1} = f'(-1)=0$ also $f''(\lambda)|_{\lambda=-1} = f''(-1)=0$ but $f'''(-1)\neq 0$

"extremal point" — critical point

Solve $(D^4 - D^3 - 9D^2 - 11D - 4)\, y = e^{2x} + e^{-x} + 3$. (V.Imp)

A.E. $f(\lambda) = \lambda^4 - \lambda^3 - 9\lambda^2 - 11\lambda - 4 = 0 \quad (\lambda+1)(\lambda^3 - 2\lambda^2 - 7\lambda - 4) = 0$

- if $\lambda = -1 \Rightarrow$ L.H.S $1+1-9+11-4 = 0$ (if it happens that it contains another factor $(\lambda+1) \Rightarrow f'(-1)=0$)

$\lambda+1$ factor (we have to know if the root is repeated)

- $f'(\lambda) = 4\lambda^3 - 3\lambda^2 - 18\lambda - 11 = 0 \Rightarrow f'(-1) = -4-3+18-11 = 0$

hence $\lambda+1$ is a factor $f'(\lambda)$, we have to show or prove that $(\lambda+1)^2$ is a factor of $f(\lambda)$. Further

- $f''(\lambda) = 12\lambda^2 - 6\lambda - 18 \quad \therefore f''(-1) = 12+6-18 = 0$

$\therefore \lambda+1$ is a factor of $f''(\lambda)$

$\therefore (\lambda+1)^3$ is a " " $f(\lambda)$.

- $f'''(\lambda) = 24\lambda - 6$ (4×6), $f'''(-1) \neq 0 \Rightarrow$

$\therefore (\lambda+1) \Rightarrow \lambda = -1$ is a 3-times repeated root.

and hence $(\lambda+1)^3$ is a factor of $f(\lambda)$

$\therefore \lambda^3 + 3\lambda^2 + 3\lambda + 1$ is a factor of $f(\lambda)$

$\therefore f(\lambda) = \lambda^4 - \lambda^3 - 9\lambda^2 - 11\lambda - 4 = (\lambda^3 + 3\lambda^2 + 3\lambda + 1)(\lambda - 4)$

$= (\lambda+1)^3 (\lambda - 4)$.

$\therefore \lambda = -1, -1, -1, 4$

$\Rightarrow y_{c.f} = (A + Bx + Cx^2) e^{-x} + A' e^{4x}$

$(D+1)^3 (D-4) = e^{2x} + e^{-x} + 3$
$\qquad\qquad\quad I_1 \quad I_2 \quad I_3$

$y_{P.I} = \dfrac{1}{(D+1)^3(D-4)} (e^{2x} + e^{-x} + 3) \Rightarrow$ ____

* $I_1 = \dfrac{e^{2x}}{3^3(-2)} = -\dfrac{e^{2x}}{54}$.

* $I_2 = \dfrac{1}{(D+1)^3}\left(\dfrac{1}{D-4} e^{-x}\right) = \dfrac{1}{(D+1)^3} \cdot \dfrac{e^{-x}}{-5}$ (by rule 7) $=$

$= e^{-x} \dfrac{1}{(D-1+1)^3} (-1/5) = -\dfrac{1}{5} e^{-x} \dfrac{1}{D^3} e^{0} = -\dfrac{x^3}{30} e^{-x}$

* $I_3 = \dfrac{1}{(D+1)^3(D-4)} 3 = 3 \dfrac{1}{(D+1)^3(D-4)} (e^0) = -\dfrac{3}{4}$

$y_{P.I} = -\dfrac{e^{2x}}{54} - \dfrac{x^3 e^{-x}}{30} - \dfrac{3}{4}$.

$y_{G.S} = (A + Bx + Cx^2) e^{-x} + A' e^{4x} - \dfrac{e^{2x}}{54} - \dfrac{x^3 e^{-x}}{30} - \dfrac{3}{4}$.

$(1-\lambda)^{-1} = 1+\lambda+\lambda^2+\lambda^3+\cdots$; $(1+\lambda)^{-1} = 1-\lambda+\lambda^2-\lambda^3+\cdots$

Solve: $(D^3+4D)y = 6\sin x + x^2$

$\lambda^3+4\lambda = 0 \Rightarrow \lambda(\lambda^2+4) = 0$

$\lambda(\lambda-2i)(\lambda+2i) = 0$

$\lambda_1 = 0$, $\lambda_{2,3} = \pm 2i$, $y_{C.F} = A + B\cos 2x + C\sin 2x$.

$y_{P.I} = \cdot$

$y_{P.I} = \frac{1}{D(D^2+4)}(6\sin x + x^2)$ ($6\sin x$: I_1, x^2: I_2)

* $I_1 = \frac{1}{D(D^2+4)} 6\sin x = 6\frac{1}{D}\cdot\frac{1}{D^2+4}\sin x =$

$\frac{6}{D}\frac{\sin x}{-(1)^2+4} = \frac{1}{3}\cdot\frac{6}{D}\sin x = 2\int \sin x\,dx = -2\cos x$.

* $I_2 = \frac{1}{D(D^2+4)} x^2 = \frac{1}{4D(1+\frac{D^2}{4})} x^2 = \frac{1}{4D}(1+\frac{D^2}{4})^{-1} x^2 =$

$\frac{1}{4D}\left[1 - \frac{D^2}{4} + \cdots\right] x^2 = \frac{1}{4D}\left[x^2 - \frac{1}{2}\right] = \frac{1}{4}\left(\frac{x^3}{3} - \frac{x}{2}\right)$.

$y_{P.I} = -2\cos x + \frac{x^3}{12} - \frac{x}{8}$, $\Rightarrow y_{G.S} = y_{C.F} + y_{P.I}$

+ Solve: $(D^4 + 8D^2 + 16)y = \cos^2 x = \frac{1}{2}(1+\cos 2x)$

* A.E $(\lambda^2+4)^2 = 0$

$\lambda = +2i, 2i \neq -2i, -2i$

Twice. Twice.

$y_{C.F} = A e^{2ix}$

$y_{C.F} = (A+Bx)\sin 2x + (A'+B'x)\cos 2x$

$y_{P.I} = \frac{1}{(D^2+4)^2}(\frac{1}{2} + \frac{1}{2}\cos 2x) = \underbrace{\frac{1}{(D^2+4)^2}\frac{1}{2}x^0}_{I_1} + \underbrace{\frac{1}{(D^2+4)^2}\frac{1}{2}\cos 2x}_{I_2}$

$I_1 = \frac{1}{2}\frac{1}{16} = \frac{1}{32}$

$\frac{1}{2}\frac{1}{(D^2+4)^2} R e^{2ix} = \frac{1}{2} R \frac{1}{(D-2i)^2(D+2i)^2} e^{2ix} =$ | $e^{i\theta} = \cos\theta + i\sin\theta$. $\cos\theta = R\,e^{i\theta}$. $\sin\theta = \mathrm{Im}\, e^{i\theta}$.

$\frac{1}{2} R \frac{1}{(D-2i)^2}\left\{\frac{1}{(D+2i)^2} e^{2ix}\right\} = \frac{1}{2} R \frac{1}{(D-2i)^2}\frac{e^{2ix}}{(2i+2i)^2} =$

$\frac{1}{32} R \frac{1}{(D-2i)^2} e^{2ix} \cdot 1 = R\left[\frac{1}{32} e^{2ix}\frac{1}{D^2}\cdot 1\right] =$

$\frac{R}{2}\left[\frac{1}{32}\frac{x^2}{2} e^{2ix}\right] = \frac{x^2}{64}\cos 2x$.

$(D^2+aD+b)y = f(x) \Rightarrow y_{G.S} = y_{C.F} + y_{P.I}$

* Solve $(D^2+4)y = \cos 3x$ / $(D^2+4)y = \cos 2x$ (R.I.mP)

Solve: $(D^3 - D^2 - 2D)y = 0$, $(D^3 - 8D^2 + 9)y = 0$

Solve: $(D^2+4)y = \sec 2x$. or $(D^2+a^2)y = \sec ax$

$\lambda = \pm 2i$, $-2i$ $\quad y_{C.F} = A\cos 2x + B\sin 2x$ $\quad y_{P.I} = \frac{x}{a}\sin ax + \frac{\cos ax}{a^2}\ln ax$

$y_{P.I} = \frac{1}{D^2+4}\sec 2x \Rightarrow$

$\frac{1}{4i}\left[\frac{1}{D-2i} - \frac{1}{D+2i}\right]\sec 2x \Rightarrow$

I_1 $\qquad$ I_2

$I_1 = \frac{1}{D-i2}\sec 2x - \frac{1}{D-2i} e^{2ix}\cdot e^{-2ix}\sec 2x =$

$e^{2ix}\frac{1}{D}(e^{-2ix}\sec(2x)) = e^{2ix}\int((\cos 2x - i\sin 2x)\sec 2x\, dx$

$= e^{2ix}\int(1 - i\tan 2x)\, dx \Rightarrow I_1 = e^{2ix}\left(x + \frac{i}{2}\ln(\cos 2x)\right)$

$I_2 = \frac{1}{D+2i} e^{2ix}\cdot e^{-2ix}\sec(2x) = \frac{1}{D+2i} e^{-2ix}(e^{2ix}\sec(2x)) =$

$e^{-2ix}\frac{1}{D-2i+2i} e^{2ix}\sec 2x = e^{-2ix}\int(\cos 2x + i\sin 2x)\sec 2x\, dx =$

$e^{-2ix}\int dx + \frac{i}{2}\int 2\tan 2x\, dx = e^{-2ix}\left(x + \frac{i}{2}\ln|\sin 2x|\right)$.

Finally, $y_{G.S} = y_{C.F} + \frac{e^{2ix}}{4i}\left(x + \frac{i}{2}\ln|\cos 2x|\right) - \frac{e^{-2ix}}{4i}\left(x + \frac{i}{2}\ln(\sin 2x)\right)$.

Solve: $(D^2 + a^2)y = \sec^3 ax$

A.Eqtn: $\lambda^2 + a^2 = 0 \Rightarrow \lambda = \frac{\pm\sqrt{-4a^2}}{2} = \pm ia \Rightarrow y_{C.F} = A\cos ax + B\sin ax$.

$y_{P.I} = \frac{1}{D^2+a^2}\sec^3 ax = \frac{1}{(D+ia)(D-ia)}\sec^3 ax =$

$\frac{1}{(D+ia)(D-ia)} e^{iax}(e^{-iax}\sec^3 ax) = \frac{1}{D+ia} e^{iax}\int e^{-iax}\sec^3 ax\, dx =$

$\frac{a}{D+ia} + \frac{b}{D-ia} = \frac{1}{2}\left(\frac{1}{D+ia} + \frac{1}{D+ia}\right) = \frac{-1}{2ia}\left(\frac{1}{D+ia} - \frac{1}{D-ia}\right) \Rightarrow$

$y_{P.I} = -\frac{1}{2ia}\left(\frac{1}{D+ia} - \frac{1}{D-ia}\right)(\sec^3 ax\cdot e^{iax}\cdot e^{-iax}) =$

I_1 $\qquad$ I_2

$I_1 = \frac{1}{D+ia} e^{-iax}\cdot(e^{iax}\sec^3 ax) = e^{-iax}\int(\cos ax + i\sin ax)\frac{1}{\cos^3 ax}\, dx =$

$e^{-iax}(\tan ax + i\int \sec ax\cdot(\sec ax\tan ax\, dx)) = \tan ax + i\int \sec ax\, d(\sec ax) =$

$e^{-iax}(\tan ax + \frac{1}{2}i\sec^2 ax)$; $I_2 = (\tan ax - \frac{1}{2}i\sec^2 ax)e^{+iax} \Rightarrow$

$y_{G.S} = y_{C.F} + \frac{i}{2a}(I_1 - I_2)$.

Bessel's Functions:

1st Consider the DE, $x^2 y'' + x y' + (x^2 - n^2) y = 0$ — (1)

Bessel D.E, which depends on a Constant n.

For such a kind of D.E's, we Can use series-Solution method to solve it.

$$y = \sum_{r=0}^{\infty} a_r x^{c+r} \Rightarrow y' = \sum_{r=0}^{\infty} (r+c) a_r x^{c+r-1}$$
$$y'' = \sum_{r=0}^{\infty} (c+r)(c+r-1) a_r x^{c+r-2} \quad \Big\} (2)$$

Subst. (2) in (1) $\Rightarrow$

$$\sum_{r=0}^{\infty} (c+r)(c+r-1) a_r x^{c+r} + \sum_{r=0}^{\infty} (c+r) a_r x^{c+r} + \sum_{r=0}^{\infty} a_r x^{c+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{c+r} = 0$$

① $c(c-1) a_0$ is the Coefficient of x^c i.e. $(r=0)$

$$c(c-1) a_0 + c a_0 - n^2 a_0 = 0$$
$$a_0 (c^2 - c + c - n^2) = a_0 (c^2 - n^2) = 0 \quad — (3)$$

Since $a_0 \neq 0 \Rightarrow c^2 - n^2 = 0 \Rightarrow \boxed{c = \pm n}$

(3) is Called the indicial eqtn

② $(r \neq 0)$ x^{c+r} : $(c+r)(c+r-1) a_r + (c+r) a_r - n^2 a_r + a_{r-2} = 0$

$a_r[(c+r)(c+r-1+1) - n^2] + a_{r-2}$ Putting $r-2$ for r in (*).

$$[(c+r)^2 - n^2] a_r = -a_{r-2}$$
$$a_r = -a_{r-2} / [(c+r)^2 - n^2]$$

$$\sum_{r=0}^{\infty} [(c+r)(c+r-1) + (c+r) - n^2] a_r x^{c+r} = \sum_{r=0}^{\infty} -a_r x^{c+r-2}$$

$$\sum_{r=0}^{\infty} [(c+r)^2 - n^2] a_r x^{c+r} = \sum_{r=0}^{\infty} -a_r x^{c+r-2}$$

$$[(c^2 - n^2) a_0] x^c = (-a_0) x^{c-2}$$
$$[((c+1)^2 - n^2) a_1] x^{c+1} = (-a_1) x^{c-1}$$
$$[((c+2)^2 - n^2) a_2] x^{c+2} = (-a_2) x^c$$
$$[((c+3)^2 - n^2) a_3] x^{c+3} = (-a_3) x^{c+1}$$

① $[(c+r)^2 - n^2] a_r (x^{c+r}) = -a_{r+2} (x^{c+r}) \quad r = 0, 1, 2, \ldots$

②

$$a_0 x^{c-2} + a_1 x^{c-1} + \sum_{r=0}^{\infty} [((c+r)^2 - n^2) a_r + a_{r+2}] x^{c+r} = 0$$

$$J_{\pm n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\pm n + r + 1)} \left(\frac{x}{2}\right)^{\pm n + 2r}$$

1st we can write two solutions for (1) in such a way,

$$a_0 = 1/2^n \Gamma(n+1)$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r}$$

only one solution exist when $n=0$

which is the solution corresponding to $C = +n$

2nd solution is Putting (n) above as $a_0 = 1/2^{(-n)}\Gamma(-n+1)$.

$$J_{-n}(x) = \sum_{r=0} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} (x/2)^{-n+2r}$$

Corresponding to $C = -n$

$J_n(x)$, $J_{-n}(x)$

$J_n(x)$ is called Bessel's fn of order (n).

In general, the general solution can be written as $y_{GS} = A J_n(x) + B J_{-n}(x)$

$$y = A J_n(x) + B J_{-n}(x)$$

$J_n(x) = \sum_{r=0}^{\infty} [(-)^r \cdot (x/2)^{n+2r} / r! \cdot \Gamma(n+r+1)]$; $J_{-n}(x) = \sum_{r=0}^{\infty} [(-)^r \cdot (x/2)^{-n+2r} / r! \cdot \Gamma(-n+r+1)]$

if $n=0$ $\Rightarrow$ we have one solution $J_0(x)$

2nd we if $n \in Z^+$ $\Rightarrow$ the two solutions depend on each other $\Rightarrow$ i.e they are related.

Prove: $J_{-n}(x) = (-)^n J_n(x)$,

$J_{-n}(x) = (-)^n J_n(x)$, $n \in Z^+$.

(24)

(Go to) Harry Bateman (1882–1946) Mathematician (pure) Father of special fⁿˢ.

Proof:- $$J_n(x) = \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma(n+s+1)} (x/2)^{n+2s}$$

Since $\Gamma(t) = \infty \quad (t \le 0)$; t integer.

Hence, $$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(-n+r+1)} (x/2)^{-n+2r}$$

$\Gamma(-n+r+1) = \infty$ (i) $-n+r+1 \le 0$

i.e at, $r \le n-1$

Therefore,

$$\frac{1}{\Gamma(-n+r+1)} = 0 \quad ; \; r \le n-1$$

therefore we write $$J_{-n}(x) \sum_{\substack{r=n \\ s=(r-n)=0}}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} (x/2)^{-n+2r}$$

to make the summation from 0 to ∞ ⇒

taking $r = n+s$ ⇒ $r-n = r+\underline{r-n} = \underline{r+s}$

since $r = n+s \Rightarrow \underline{n+2s}$

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-)^{n+s}}{(n+s)!\,\Gamma(s+1)} (x/2)^{2n+2s-n} =$$

$$(-)^n \sum_{s=0}^{\infty} \frac{(-)^s}{\Gamma(n+s+1)\,s!} (x/2)^{n+2s} =$$

$$(-)^n \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma(n+s+1)} (x/2)^{n+2s}$$ which is

If n is an integer we obtain a unique solution.

i.e the two solutions are not independent.

* we want to know when the two solutions of B.D.E independent.

3rd

Recurrence Relations :-

$$D(x^{\pm n} J_{\pm n}) = \pm x^{\pm n} J_{n\mp1}$$

$d(x^n J_n(x))/dx = x^n J_{n-1}(x)$.

$d(x^{-n} J_n(x))/dx = -x^{-n} J_{n+1}(x)$.

* $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ (1)

* $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$ (2)

[Dr./Samir Anies 917788 (Homer)]

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r} \Rightarrow$$

Cancelled there

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r} =$$

$$2^n (x/2)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r} = 2^n \sum_{0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{2n+2r}$$

$$\Rightarrow \frac{d}{dx}(x^n J_n(x)) = 2^n \sum_{r=0}^{\infty} \frac{(-)^r (2n+2r)}{r!\,\Gamma(n+r+1)} (x/2)^{2n+2r-1} \times (1/2) =$$

$$2^n \sum_{0}^{\infty} \frac{(-)^r (n+r)}{r!\,(n+r)\,\Gamma(n+r)} (x/2)^{2n+2r-1} =$$

$$\left(\frac{x}{2}\right)^n \cdot \left(\frac{x}{2}\right)^{n+2r-1} = \left(\frac{x}{2}\right)^n \cdot \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$2^n (x/2)^n \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma((n-1)+r+1)} (x/2)^{(n-1)+2r} =$$

$$x^n \sum_{r=0}^{\infty} \frac{(-)^r (x/2)^{(n-1)+2r}}{r!\,\Gamma((n-1)+r+1)} = x^n J_{n-1}(x)$$

* $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$.

proof:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r} \Rightarrow$$

Cancelled here

$$x^{-n} J_n(x) = x^{-n} (x/2)^n \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{2r} =$$

$$2^{-n} \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{2r} .$$

$$\frac{d}{dx}(x^{-n} J_n(x)) = 2^{-n} \sum_{r=1}^{\infty} \frac{(-)^r (2r)}{r!\,\Gamma(n+r+1)} (x/2)^{2r-1} \cdot 1/2 =$$

$$2^{-n} \sum_{r=1}^{\infty} \frac{(-)^r}{(r-1)!\,\Gamma(n+r+1)} (x/2)^{2r-1}$$ Set $r = 1+s \Rightarrow$

put $(r-1) = s \Rightarrow 2r-1 = r+r-1 = r+s = 1+s+s = 2s+1$

$$2^{-n} \sum_{s=0}^{\infty} \frac{(-)^{s+1}}{s!\,\Gamma((n+1)+s+1)} (x/2)^{2s+1} =$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma((n+1)+s+1)} (x/2)^{(n+1)+2s} .$$

$$= -x^{-n} J_{n+1}(x) .$$

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$; $2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

From (1), (2) before =>

$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$ (3) / (x^n)

$x^{-n} J_n'(x) + n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$ (4) $\times x^n$

* $J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$ (5) = (3)'

* $J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$ (6) = (4)'

(5) + (6) =>

* $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ (7)

(5) − (6) =>

* $2\frac{n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$ (8)

$\sqrt{\frac{2}{\pi x}} = e^y \Rightarrow \frac{1}{2}\ln\left(\frac{2}{\pi x}\right) = y \Rightarrow$

$\sqrt{2/\pi x}\,[\cos x + i \sin x] = J_{-1/2}(x) + iJ_{1/2}(x) = \sqrt{\frac{2}{x\pi}}\, e^{ix} = \exp\left(\frac{1}{2}\ln\frac{2}{\pi x} + ix\right)$

4th Show that:

$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$.

$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

$\sin^2 + \cos^2 = 1$

$J_{1/2}^2(x) + J_{-1/2}^2(x) = \frac{2}{\pi x}$

$\sin x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$, $\cos x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2r!}$.

Since $\sum_{r=0} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$, $n = 1/2$ =>

$J_{1/2}(x) = \sum_{r=0} \frac{(-)^r (x/2)^{2r+1/2}}{r!\,\Gamma(r+3/2)}$ $\Gamma(r+\frac{1}{2}+1) = (r+\frac{1}{2})\,\Gamma(r+\frac{1}{2})$, $\Gamma(r+\frac{1}{2})$

but $\Gamma(r+\frac{3}{2}) = (r+\frac{1}{2})(r-\frac{1}{2})\ldots\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(1/2)$

$(r+1)$ factor

$\Gamma(r+\frac{3}{2}) = \frac{1}{2^{r+1}}[(2r+1)(2r-1)\cdots 3\cdot 1]\sqrt{\pi}$

$\Gamma(r+\frac{3}{2}) = \frac{r(r-1)\cdots 2\cdot 1\,(2r+1)(2r-1)\cdots 3\cdot 1}{2^{r+1}}\sqrt{\pi} =$

$\frac{2r(2r-2)\cdots 4\cdot 2\times(2r+1)(2r-1)\cdots 3\cdot 1}{2^{2r+1}}\sqrt{\pi} =$

$\frac{(2r+1)!}{2^{2r+1}}\sqrt{\pi}$. Substit. in $J_{1/2}(x)$:

$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-)^r 2^{2r+1}}{(2r+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2r+\frac{1}{2}} = \times \frac{\sqrt{x/2}}{\sqrt{x/2}}$

$\sqrt{\frac{2}{\pi x}} \sum_{r=0}^{\infty} \frac{(-1)^r 2^{2r+1}}{(2r+1)!\,2^{2r+1}} x^{(2r+1)} = \sqrt{\frac{2}{\pi x}} \sin x$.

Taking $n = -1/2$ in $J_n(x)$ $\Rightarrow$

$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1/2)} \cdot (x/2)^{2r-1/2}$

$r!\,\Gamma(r+1/2) = (r-\tfrac{1}{2})\cdot r!\,\Gamma(r-\tfrac{1}{2})$

Another method: Using Multiplication-theorem of Gauss and Legendre when $m=2$.

$2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$ i.e. if $x = r+1/2$ $\Rightarrow$

$2^{2r}\Gamma(r+1/2)\cdot\Gamma(r+1) = \sqrt{\pi}\,\Gamma(2r+1) =$

$2^{2r}\Gamma(r+1/2)\,r! = \sqrt{\pi}\,(2r!)$ $\Rightarrow$

$r!\,\Gamma(r+1/2) = \frac{\sqrt{\pi}\,(2r)!/2^{2r}}{}$. $\Rightarrow$ $J_{-1/2} = \sum_{r=0}^{\infty} (-)^r (x/2)^{n+2r}/r!\,\Gamma(n+r+1)$ $(n=-1/2)$

$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{2r}}{\sqrt{\pi}\,2r!}\cdot(x/2)^{2r-1/2} = \sqrt{\frac{2}{\pi x}} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma}{2r!} \frac{2^{2r}}{(x/2)^{2r}} =$

$\Rightarrow J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2r!} = \sqrt{\frac{2}{\pi x}}\cdot \cos x$

(28)

$$x^{m+r} J_{m+r}(x) = (-1)^r \left(\frac{d}{x\,dx}\right)^r (x^m J_m(x)) \quad , n \in \mathbb{Z}$$

Calculate $J_{3/2}(x)$, $J_{-3/2}(x)$.

From 7, 8 (before)

when $n = 1/2$ → in (8)

$$\frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x \Rightarrow$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right) .$$

$$2 J'_{-1/2}(x) = J_{-3/2}(x) - J_{1/2}(x) \Rightarrow$$

$$J_{-3/2}(x) = 2 J'_{-1/2}(x) + J_{1/2}(x) = 2\sqrt{\frac{2}{\pi}} \frac{d}{dx}\left(\frac{\cos x}{\sqrt{x}}\right)$$

$$2\sqrt{\frac{2}{\pi}}\left[x^{-1/2} \cdot \sin x - \frac{1}{2}\cos x \cdot x^{-3/2}\right] + \sqrt{\frac{2}{\pi x}} \cdot \sin x .$$

$$\sqrt{\frac{2}{\pi}}\left[-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x\sqrt{x}}\right] = \sqrt{\frac{2}{\pi x}}\left[-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x\sqrt{x}}\right]$$

odd odd

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)$$

always appear

- Calculate in terms of x , $\sin x$, $\cos x$ each of

$J_{5/2}(x)$, $J_{-5/2}(x)$.

5th

In general $J_{m+1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot (-1)^m x^{m+1/2} \left(\frac{d}{x\,dx}\right)^m \left(\frac{\sin x}{x}\right)$

and m is a (+ve) integer.

Special Case : $m = 1$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi}} (-1)^1 x^{3/2} \left(\frac{d}{x\,dx}\right)\left(\frac{\sin x}{x}\right) =$$

$$= -\sqrt{\frac{2}{\pi}} \cdot x^{1/2} \frac{x\cos x - \sin x}{x^2} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

$$J_{r+n}(x) = (-1)^r x^{r+n} \left(\frac{d}{x\,dx}\right)^r (x^{-n} J_n(x))$$

$$J_{m+\frac{1}{2}}(x) = (-1)^m \sqrt{\frac{2}{\pi}}\, x^{m+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^m \left[\frac{\sin x}{x}\right]$$

In general $J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \cdot (-1)^m x^{m+\frac{1}{2}} \cdot \left(\frac{d}{x\,dx}\right)^m (\sin x / x)$. $m \in Z^+$

Proof: From previously the eq^n (By Induction)

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \quad \text{. Made is symmetry}$$

$$\frac{d}{dx}\left(x^{(n+1)} J_{n+1}(x)\right) = \left(\frac{d}{x\,dx}\right)\left(x^{-n} J_n(x)\right) = -x^{-(n+1)} J_{n+2}(x)$$

$$\frac{d}{x\,dx}\left(-\left(\frac{d}{x\,dx}\right)\left(x^{-n} J_n(x)\right)\right) = \frac{1}{x} \times -x^{-(n+1)} J_{n+2}(x) = -x^{-(n+2)} J_{n+2}(x)$$

$$= -\left(\frac{d}{x\,dx}\right)^2 \left(x^{-n} J_n(x)\right)$$

applying this eq^n twice

notice $K = d/x\,dx \Rightarrow KK = (K)^2$

$x^{-(n+2)} J_{n+2}(x)$

$$x^{(n+3)} J_{n+3}(x) = (-)^3 \left(\frac{d}{x\,dx}\right)^3 \left(x^{-n} J_n(x)\right) .$$

In general for (+ve) integer (m), we have

$$x^{-(n+m)} J_{n+m}(x) = (-)^m \left(\frac{d}{x\,dx}\right)^m \left(x^{-n} J_n(x)\right)$$

((m is (+ve) integer)) , n = Const. as $n = 1/2$

$$x^{-(\frac{1}{2}+m)} J_{1/2+m}(x) = (-)^m \left(\frac{d}{x\,dx}\right)^m \left(x^{-1/2} J_{1/2}(x)\right) .$$

Since $x^{-1/2} J_{1/2}(x) = x^{-1/2} \sqrt{\frac{2}{\pi x}} \sin x = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin x}{x}$

$$J_{m+1/2}(x) = (-)^m \sqrt{\frac{2}{\pi}} \cdot x^{m+1/2} \left(\frac{d}{x\,dx}\right)^m \left(\frac{\sin x}{x}\right) .$$

apply this formula to find $J_{5/2}$ i.e $m = 2$

$$\boxed{x^{-(n+r)} J_{n+r} = (-1)^r \left[\frac{D}{x}\right]^r \left(x^{-n} J_n\right)}$$

$r = m$

Euler's (Homogeneous) differential Equation

1st General Form : $\left(\sum_{k=0}^{n} a_k x^k D^k\right) y = 0$

("homogeneity comes from $x^k D^k$")

$$\left(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \cdots + a_1 x D + a_0\right) y = 0$$

It is noticed, the coefficient of any term in D contains the x for the same power

This eqtn can be converted into a L.DE in Constant Coefficients

$x = e^t \Rightarrow \frac{d}{dx} = \frac{d}{dt}\cdot\frac{dt}{dx} \Rightarrow t = \ln x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

* $D = d/dx$, $\theta = d/dt$. $\Rightarrow D = \frac{dt}{dx}\cdot\frac{d}{dt}$

(the substitution disappeared) $xD = \theta \Rightarrow (xD) = (\theta)$ [the previous relation becomes]

$D^2 \Rightarrow D^2 \quad x^2 D^2 = \theta(\theta-1)$ (it can be proved that)

$x^3 D^3 = \theta(\theta-1)(\theta-2)$

$\vdots$

$$x^n D^n = \prod_{K=1}^{n} (\theta - K + 1)$$

$x^n D^n = \theta(\theta-1)(\theta-2) \cdots (\theta-(n-1))$

~ ~ ~

Solve: $(x^2 D^2 - 4xD + 6)y = x^5$.

$x = e^t$, $\theta = \frac{d}{dt}$, $xD = \theta$; $x^2 D^2 = \theta(\theta-1)$

$[\theta(\theta-1) - 4\theta + 6]y = e^{5t}$

$(\theta^2 - 5\theta + 6)y = e^{5t}$ L.D.E. in Const. Coefficients.

A.E : $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0$; $\lambda_{1,2} = 2, 3$

$y_{C.F} = A e^{2t} + B e^{3t}$ $\Rightarrow$ $y_{P.I} = \frac{1}{\theta^2 - 5\theta + 6} e^{5t}$ =

$\frac{e^{5t}}{25 - 25 + 6} = \frac{e^{5t}}{6}$ $\Rightarrow$ $y_{G.S} = A e^{2t} + B e^{3t} + \frac{e^{5t}}{6}$ =

$y_{G.S} = A x^2 + B x^3 + \frac{x^5}{6}$.

~ ~ ~

$x^2 \frac{d^2}{dx^2} = x^2 D^2 = x^2[D\cdot D] = x^2\left[D\frac{x\,dt}{dx}\cdot\frac{d}{dt}\right] = x^2[D\frac{\theta}{x}] =$

$x^2[D\frac{d\theta}{dx}\cdot\frac{dt}{dt}] = x^2[D(x^{-1}\theta)]$

$x^2[-x^{-2}\theta + x^{-1} D\theta] = x^2\left[-x^{-2}\theta + x^{-1}\frac{dt}{dx}\frac{d}{dt}\cdot\frac{d}{dt}\right] =$

$x^2[-x^{-2}\theta + x^{-2}\theta^2] = \theta^2 - \theta = \theta(\theta-1)$.

Solve the eqtn : $(x^3D^3 - x^2D^2 + 2xD - 2)y = x^3 + 3x$

Let $x = e^t$, $t = \ln x$, $\theta = \frac{d}{dt}$

$xDy = \theta y$; $x^2D^2y = \theta(\theta-1)y$

$x^3D^3y = \theta(\theta-1)(\theta-2)y$

$\theta(\theta^2 - 3\theta + 3) = \theta^3 - 3\theta^2 + 3\theta$, $\theta^2 - \theta$ =>

$\theta^3 - 3\theta^2 + 3\theta^2 + \theta^2 - \theta$ -

$$[\theta(\theta-1)(\theta-2) - \theta(\theta-1) + 2\theta - 2]y = e^t + 3e^t$$

$$(\theta-1)(\theta^2 - 3\theta + 2)y = e^{3t} + 3e^t$$

which is a linear eqtn in Constant Coefficients.

1st A.E. : $(\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$

$(\lambda-1)(\lambda-2)(\lambda-1) = 0$ => $\lambda = 2, 1, 1$ =>

$$y_{C.F} = (A + Bt)e^t + Ce^{2t}$$ =>

$$y_{P.I} = \frac{1}{(\theta-1)^2(\theta-2)}(\underset{I_1}{e^{3t}} + \underset{I_2}{3e^t})$$

$$I_1 = e^{3t}\frac{1}{4+1} = \frac{e^{3t}}{4}$$

$$I_2 = \frac{1}{(\theta-1)^2(\theta-2)}(3e^t) = 3\frac{1}{(\theta-1)^3}\cdot\frac{e^t}{1^3 - 2} =$$

$$\frac{1}{(\theta-1)^2} - 3e^t = e^t\frac{1}{(\theta+1-1)^2} = 3 = e^t \int t^{-2}\,dt$$

$$e^t\left(\int t^{-2}\,dt\right) = e^t \times \frac{-3}{2}t^2$$ =>

$$y_{G.S} = (A + Bt)e^t + Ce^{2t} + \frac{e^{3t}}{4} - \frac{3}{2}t^2e^t -$$

$$(A + B\ln x)x + Cx^2 + \frac{x^3}{4} - \frac{3}{2}x(\ln x)^2$$

Commetary: By doing that, we have absorbed the Power of (x) which lie side to D, or the other Powers x^n side to D^n. The eqtn $(\sum_{i=0}^{n} a_i x^i D^i)y = 0$ Can be transformed into the L.DE eqtn with Constant Coefficients ; $x = e^t$, $t = \ln x$; finding the solution we substitute t by $\ln x$.

Generalized Euler Equation : $[\sum_{k=0}^{n} a_k (a+bx)^k D^k]y = F(x)$
"which is a homogeneous L.DE"

2nd Generalized Euler Equation :- $[\sum_{k=0}^{n} a_k (a+bx)^k D^k]y = F(x)$

$a_n (a+bx)^n D^n + a_{n-1}(a+bx)^{n-1} D^{n-1} + \cdots$

$a_1 (a+bx) D + a_0 = F(x)$

$a+bx = e^t$, $\theta \equiv \frac{d}{dt}$, $D \equiv \frac{d}{dx}$

$\ln(a+bx) = t = \ln(a+bx)$,

$\frac{d}{dt}\quad \frac{dt}{dx} = \frac{b}{a+bx}$

$D \equiv \frac{dt}{dx}\cdot\frac{d}{dt} \equiv \frac{dt}{dx}\theta \Rightarrow$

$[1/(dt/dx)]D \equiv \theta$

$\therefore \frac{d}{dt}y = \frac{dy}{dx}\cdot\frac{dx}{dt} = \theta y = Dy \cdot \frac{a+bx}{b}$

$\frac{(a+bx)}{b} Dy = \theta y \Rightarrow$

$(a+bx) Dy = b\theta y$

$(a+bx)^2 D^2 y = b^2 \theta(\theta-1) y$

$(a+bx)^3 D^3 y = b^3 \theta(\theta-1)(\theta-2) y$

In General $(a+bx)^n D^n y \equiv$

$b^n \theta(\theta-1)(\theta-2)\cdots(\theta-(n-1)) y$

where b is Coefficient of x.

$(a+bx)^n D^n \equiv [\prod_{k=1}^{n} b(\theta-k+1)] \equiv b^n [\prod_{k=1}^{n} (\theta-k+1)]$

$D^2 \equiv D\cdot D \equiv D(\frac{1}{a+bx}\cdot(a+bx)D) \equiv$

$D\frac{1}{a+bx}(a+bx)D \equiv D\frac{1}{a+bx} b\theta \equiv$

$D(\frac{b}{a+bx}\theta) \equiv$

$-b(a+bx)^{-2} b\theta + b(a+bx)^{-1} D\theta$

since $D\theta \equiv \frac{dt}{dx}\cdot\frac{d}{dt}\cdot\frac{d}{dt} \equiv$

$b(a+bx)^{-1}\theta^2 \Rightarrow D^2 \equiv$

$-b^2(a+bx)^{-2}\theta + b^2(a+bx)^{-2}\theta^2 \Rightarrow$

$(a+bx)^2 D^2 \equiv b^2(-\theta+\theta^2)$

$\equiv b^2\theta(\theta-1)$

Solve the DE. $[(x+1)^4 D^4 + 3(x+1)^3 D^3 - 5(x+1)^2 D^2]y = \cos[\ln(x+1)^2]$.

$x+1 = e^t \Rightarrow t = \ln(x+1)$,

$\frac{d}{dt} \equiv \theta$,

$(x+1)^2 D^2 \equiv 1^2\theta(\theta-1)$

$(x+1)^3 D^3 \equiv 1^3\theta(\theta-1)(\theta-2)$

$(x+1)^4 D^4 \equiv 1^4\theta(\theta-1)(\theta-2)(\theta-3)$

$b = 1$

$[\theta(\theta-1)(\theta-2)(\theta-3)+3\theta(\theta-1)(\theta-2)-5(\theta)(\theta-1)]y = \cos 2t$

$\theta(\theta-1)(\theta^2-5\theta+6+3\theta-6-5)y = \cos 2t$

$\theta(\theta-1)(\theta^2-2\theta-5)y = \cos 2t$

the discriminant $b^2-4ac = 4+20 = 24$

with fractional numbers; Can not be factorized in terms of a Complete square.

A.E : $\lambda(\lambda-1)(\lambda^2-2\lambda-5)=0$ $\quad \frac{2\pm\sqrt{24}}{2}$ $\quad 1\pm\sqrt{6}$

$\lambda = 0, 1, 1+\sqrt{6}, 1-\sqrt{6}$ $\Rightarrow$

$y_{C.F} = A + Be^t + A'e^{(1+\sqrt{6})t} + B'e^{(1-\sqrt{6})t}$

$y_{P.I} = \frac{1}{\theta(\theta-1)(\theta^2-2\theta-5)}\cdot\cos 2t =$

$\frac{1}{(\theta^2-\theta)(\theta^2-2\theta-5)}\cdot\cos 2t =$

$\frac{1}{(-4-\theta)(-4-2\theta-5)}\cos 2t = \frac{1}{(\theta+4)(2\theta+9)}\cos 2t$

$\frac{1}{2\theta^2+17\theta+36}\cos 2t = \frac{1}{-8+17\theta+36}\cos 2t =$

$\frac{28-17\theta}{28-17\theta}\cdot\frac{1}{17\theta+28}\cos 2t = (28-17\theta)\cdot\frac{1}{(28)^2+289\times 4}\cos 2t$

we multiplied by the Conjugate

$\frac{1}{1940}\,28\cos 2t + \underline{17}\sin 2t$

$y = A + B(1+x) + A'(1+x)^{1+\sqrt{6}} + B'(1+x)^{(1-\sqrt{6})} +$

$\frac{28}{1940}\cos 2\ln(x+1) + \frac{34}{1940}\sin 2\ln(x+1)$.

Solve: $(D^2-2D+1)y = x(e^x)(\sin x)$

A.Eqn $\lambda^2-2\lambda+1=0 \Rightarrow (\lambda-1)^2=0 \Rightarrow \lambda = 1,1$

$y_{C.F} = (A+Bx)e^x$.

(34)

~~$(D^4 + 5D^2 + 4)y = e^x + e^{-x}$~~

~~A.E~~ $\Rightarrow m^4 + 5m^2 + 4 = 0$ $\Rightarrow$ ~~$m = \pm i$, $m = \pm 2i$~~

$$y_{P.I} = \frac{1}{(D-1)^3} x e^x \sin x = \frac{1}{(D-1)^3} e^x (x \sin x) =$$

$$e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D}\left(\frac{1}{D} x \sin x\right) =$$

$(\int u\,dv = uv - \int v\,du)$

$$e^x \frac{1}{D} \int x \sin x\,dx = e^x \frac{1}{D}\left[-x\cos x + \int \cos x\,dx\right] =$$

$$e^x \frac{1}{D}\left[-x\cos x + \sin x\right] =$$

$$e^x \int \left[-x\cos x + \sin x\right] dx$$

$$e^x \left[-\int x\, d(\sin x) - \int \cos x\right] =$$

$$e^x \left[-x\sin x + \int \sin x\,dx - \cos x\right] = e^x\left[-x\sin x + 2\cos x\right]$$

$$\Rightarrow -e^x (x\sin x + 2\cos x) = y_{P.I}$$

$$y_{G.S} = (A + Bx)e^x - e^x(x\sin x + 2\cos x)$$

Solve: $(D^2 + a^2)y = \sec^3 ax$

1st Simultaneous 2.D eqtns with Constant Coefficients :

Consider; $f(D)\,x + \varphi(D)\,y = F(t)$ ①

$\chi(D)\,x + \psi(D)\,y = G(t)$ ②

where $D = \frac{d}{dx}$ and the independent variable is "t"

$f(D)$, $\varphi(D)$, $\psi(D)$, $\chi(D)$ are Polynomials of D

In order to omit(y) :

$\chi(D)$ ① — ③ and $\varphi(D)$ ② — ④, i.e $\chi(D)$ operates on ①, and $\varphi(D)$ operates on ② ⇒ (I) − (II) results in,

$$[\chi(D) f(D) - \varphi(D)\psi(D)]\,x = \chi(D) F(t) - \varphi(D) G(t)$$

$$F(D)\,x = R(t) \quad ⑤$$

where $F(D) = \chi(D) f(D) - \varphi(D)\psi(D)$

$R(t) = \chi(D) F(t) - \varphi(D) G(t)$

and ⑤ Can be solved by the methods developed to solve such DE's.

$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_n e^{\lambda_n t} + P(t)$

$P.I = P(t) = \frac{1}{F(D)} R(t)$

In order to omit (x) : $\psi(D)$ ① — ③, $f(D)$ ② — ④ ⇒ (II) − (I)

$$[\chi(D) f(D) - \varphi(D)\psi(D)]\,y = f(D) G(t) - \psi(D) F(t)$$

$$F(D)\,y = R^*(t) \quad (4)$$

It is noticed that $F(D)$ is the same i.e we get the same A.eqtn ⇒ same C.F

$y = B_1 e^{\lambda_1 t} + B_2^* e^{\lambda_2 t} + \cdots + B_n e^{\lambda_n t} + P^*(t)$

$P^*(t) = P.I = \frac{1}{F(D)} R^*(t)$

It is noticed that we obtained $n+n = 2n$ Constant, not n Constant.

Now substituting both of x and y that we have already obtained in either ; the appropriate, ① or ②, we can find the relation between the A's and the B's, therefore we get n Constants only.

$$\begin{bmatrix} f(D) & \varphi(D) \\ \psi(D) & \chi(D) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F(t) \\ G(t) \end{bmatrix} \Rightarrow$$

$$\frac{x}{\begin{vmatrix} F(t) & \varphi(D) \\ G(t) & \chi(D) \end{vmatrix}} = \frac{y}{\begin{vmatrix} f(D) & F(t) \\ \psi(D) & G(t) \end{vmatrix}} = \frac{1}{\begin{vmatrix} f(D) & \varphi(D) \\ \psi(D) & \chi(D) \end{vmatrix}} \quad \text{i.e } \frac{x}{R(t)} = \frac{y}{R^*(t)} = \frac{1}{F(D)}$$

Solve the S. L. DE $(D^2+5)y - 4z = -36\cos(7x)$ (1)

$y + D^2 z = 99\cos(7x)$ (2)

$D = \frac{d}{dx}$; the Independent variable is x . To omit z

D^2 (1) — (I)

4 (2) — (II) $\Rightarrow$ (I) + (II) =

$[D^2(D^2+5)+4]y = -36D^2\cos 7x + 4\times 99\cos(7x)$

$= -36(-7^2)\cos(7x) + 4\times 99\cos(7x) \Rightarrow$

$[(D^2+1)(D^2+4)]y = 36(49\cos 7x + 11\cos(7x)) = 36\cos(7x)\times 60$

$\Rightarrow$ A eqth is $\lambda = \pm i, \lambda = \pm 2i \Rightarrow$

$y_{C.F} = A_1\cos x + A_2\sin x + A_3\cos x + A_4\sin x$.

$y_{P.I} = \frac{1}{(D^2+1)(D^2+4)} 2160\cos 7x =$

$2160\frac{1}{(D+i)(D-i)(D+2i)(D-2i)}\cos 7x =$

$\frac{2160}{(-7^2+1)(} = \frac{2160}{2160}\cos(7x)$

$y_{G.S} = A_1\cos x + A_2\sin x + A_3\cos 2x + A_4\sin 2x + \cos(7x)$ (3)

" to omit y "

(D^2+5) (2) — (III) $\Rightarrow$ (III) — (1) =

$(D^2(D^2+5)+4)z = (D^2+5)(99\cos 7x) + 36\cos 7x$

$(D^4+5D^2+4)z = -4320\cos(7x)$

$-4320\cos(7x)$

$z_{C.F} = B_1\cos x + B_2\sin x + B_3\cos 2x + B_4\sin 2x +$

$z_{P.I} = -\frac{1}{(D^2+1)(D^2+4}\,4320\cos 7x = -\frac{4320}{2160}\cos(7x) = -2\cos 7x$

$z_{G.S} = z_{C.F} - 2\cos 7x$ (4)

To obtain the relation between the A's and the B's, we substitute (3), (4) in (2) ; for more easier calculations .

$A_1 \cos x + A_2 \sin x + A_3 \cos 2x + A_4 \sin 2x + \cos 7x$

$-B_1 \cos x - B_2 \sin x + 4B_3 \cos 2x - B_4 \sin 2x + 2x 49 \cos 7x = 99 \cos 7x$,

$(A_1 - B_1)\cos x + (A_2 - B_2)\sin x + (A_3 - 4B_3)\cos 2x + (A_4 - 4B_4)\sin 2x = 0$

$\therefore A_1 - B_1 = 0 \Rightarrow A_1 = B_1$; $A_3 - 4B_3 = 0 \Rightarrow A_3/4 = B_3$

$A_2 - B_2 = 0 \Rightarrow A_2 = B_2$; $A_4 - 4B_4 = 0 \Rightarrow A_4/4 = B_4$

Can we make the solution more simple (in Trig)

This is the general method for solving such a kind of problems. But for this particular we can follow the route quicker to solve this problem in particular.

$y = A_1 \cos x + A_2 \sin x + A_3 \cos 2x + A_4 \sin 2x + \cos 7x$

$Z = A_1 \cos x + A_2 \sin x + \frac{1}{4}A_3 \cos 2x + \frac{1}{4}A_4 \sin 2x = \frac{1}{2}\cos 7x$

Z can be found directly in this particular problem.

$4Z = (D^2 + 5)y + 36 \cos 7x$

$4A_1 \cos x + 4A_2 \sin x + A_3 \cos 2x + A_4 \sin 2x + (49 + 5)\cos 7x + 36 \cos 7x$

Dividing by (4) we get

$Z = A_1 \cos x + A_2 \sin x + \frac{1}{4}A_3 \cos 2x + \frac{1}{4}A_4 \sin 2x - 2\cos 7x$.

Solve: $(D^2 + 4)x + y = \sin^2 t$

$(D^2 + 1)y - 2x = \cos^2 t$

$\frac{dx}{dt} + 4x + 3y = t$

$\frac{dy}{dt} + 2x + 5y = e^t$

Solve:

$(D^4 + 4)y = 1$

$(D^2 + a^2)y = \sec x$ ((sec, cosec, tan, cot))

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

6th Generating function of Bessel's equation.

The function $e^{\frac{x}{2}(u-1/u)}$ is expanded exponentially in terms of powers of u

$e^{\varphi} = \sum_{n=0}^{\infty} \frac{\varphi^n}{n!}$, $\varphi = \frac{x}{2}(u - \frac{1}{u})$

$$e^{x/2(u-1/u)} = \sum_{n=-\infty}^{+\infty} u^n J_n(x) = e^X \quad , X = \frac{x}{2}(u-1/u)$$

The f^n $\exp[(x/2)(u-1/u)]$ is called the "generating function"

$$\Rightarrow e^X = \sum_{r=0}^{\infty} X^r / r! = \sum_{r=0}^{\infty} x^r (u-1/u)^r / 2^r \cdot r!$$

* notice $(u - \frac{1}{u})^r$ is expanded binomially, while r is a positive integer. "$r \in \mathbb{Z}^+$ and the zero."

$J_n(x)$ is the coefficient of u^n in the expansion

Example: Prove that: $e^{ix\sin\theta} = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta + 2i \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1)\theta$

And from here we can deduce that:

1. $1 = J_0(x) + 2\sum_{1}^{\infty} J_{2n}(x)$
2. $x = 2\sum_{0}^{\infty} (2n+1) J_{2n+1}(x)$
3. $\frac{1}{2} x \sin x = 4J_2(x) - 16J_4(x) + 36J_6(x) + \cdots$
4. $\frac{1}{2} x \cos x = J_1(x) - 9J_3(x) + 25J_5(x) - \cdots$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ 2) taking $e^{i\theta} = u$ 2)

$$e^{(ix\sin\theta)} = e^{[ix(\frac{u-1/u}{2i})]} = e^{[\frac{x}{2}(u-1/u)]}$$

$$\sum_{n=-\infty}^{-1} u^n J_n(x) + J_0(x) + \sum_{m=1}^{\infty} u^m J_m(x) \quad ; u = e^{i\theta}$$

$$\Rightarrow e^{ix\sin\theta} = \sum_{m=-\infty}^{-1} u^m J_m(x) + J_0(x) + \sum_{m=1}^{\infty} J_m(x) \cdot u^m .$$

taking the even together and the odds together

$$e^{ix\sin\theta} = J_0(x) + \sum_{1}^{\infty} (e^{2in\theta} J_{2n}(x) + e^{-2in\theta} J_{-2n}(x)) + \sum_{0}^{\infty} (e^{i(2n+1)\theta} J_{2n+1}(x) + e^{-i(2n+1)\theta} J_{-(2n+1)}(x)) .$$

Since, $J_{-n}(x) = (-1)^n J_n(x)$

$\therefore J_{-1}(x) = (-1)^1 J_1(x)$; $J_{-2n}(x) = (-1)^{2n} J_{2n}$

$J_{-(2n+1)}(x) = -J_{2n+1}(x)$ } -(2)

$\therefore e^{ix\sin\theta} = J_0(x) + \sum_1^\infty (e^{2in\theta} + e^{-2in\theta}) J_{2n}(x) + \sum_0^\infty (e^{i(2n+1)\theta} - e^{-i(2n+1)\theta}) J_{2n+1}(x) =$ (3)

i.e substituting from (2) in (1) and and solving the result.

$\therefore e^{ix\sin\theta} = J_0(x) + 2\sum_1^\infty \cos 2n\theta\, J_{2n}(x) + 2i \sum_0^\infty \sin(2n+1)\theta\, J_{2n+1}(x)$

$= \cos(x\sin\theta) + i\sin(x\sin\theta)$ (5)

i.e the reals are grouped together and the same for the imaginary parts.

$\cos(x\sin\theta) = J_0(x) + 2\sum_1^\infty \cos(2n\theta)\, J_{2n}(x)$ (4)

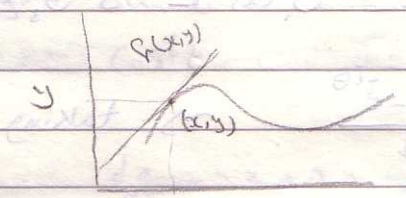
$\sin(x\sin\theta) = 2\sum_0^\infty J_{(2n+1)}(x) \sin(2n+1)\theta$. taking $\theta = 0$ in (4)

differentiating (5) w.r to θ . i.e the derivative of (5) w.r to θ.

* When J_n and J_{-n} become two independent solutions ?

that is to say differentiating twice and equating each time to the reals & / the imaginaries

$\frac{dy}{dx} = f(x,y)$



The Wronskian.

Considering this linear DE,

$y'' + f(x)y' + g(x)y = 0$ ... (1) suppose it has two solutions $y = y_1$ and $y = y_2$. When y_1 and y_2 be independent solutions?

* Since y_1 is a solution $\Rightarrow$

$\therefore\ y_1'' + f(x)y_1' + g(x)y_1 = 0$ ... (2)

* Also y_2 is a solution $\Rightarrow$

$y_2'' + f(x)y_2' + g(x)y_2 = 0$ ... (3)

* $\begin{vmatrix} y_2 & y_1 \\ (2) & (3) \end{vmatrix} =$

[1] $y_2 \times (2) - y_1 \times (3) \Rightarrow$

$y_1''y_2 - y_2''y_1 + f(x)(y_1y_2' - y_2'y_1) = 0$ ... (4)

[2] Let $\Delta = y_1'y_2 - y_2'y_1 = \Delta(y_1, y_2)$ ... (5) $\begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix}$

$\frac{d}{dx}(\Delta(y_1, y_2)) = y_1''y_2' + y_1'y_2' - (y_2'y_1' + y_2''y_1) =$

$y_1''y_2 - y_2''y_1$

Therefore (4) becomes $\frac{d\Delta}{dx} + f(x)\Delta = 0$

$\frac{d\Delta}{\Delta} = -f(x)\,dx \quad \therefore \ln\frac{\Delta}{A} = -\int f(x)\,dx$

where A is a constant of integration.

$\therefore\ \Delta = Ae^{-\int f(x)\,dx}$

* If $\Delta = 0 \Rightarrow y_1'y_2 = y_2'y_1$ $\left[= y_1y_2 \right]$

$\frac{y_1'}{y_1} = \frac{y_2'}{y_2} \Rightarrow$

$\ln y_2 = \ln(Cy_1) \quad \therefore$ " $y_2 = Cy_1$ "

* which shows the case in which the two solutions y_1, y_2 be dependent.

* Conversely if y_2 and y_1 are two dependent solutions we can show that $\Delta = 0$. A necessary and sufficient

* Applying this test to Bessel's D.E $\Rightarrow$

$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$. Considering Δ, where $A = 1$ without loss of generality.

$\therefore\ \Delta = Ae^{-\int f(x)\,dx} = Ae^{-\int \frac{dx}{x}} = \frac{A}{x}$

$\Delta(J_n, J_{-n}) = \frac{A}{x}$, What is the A here?

$\Delta(J_n, J_{-n}) = J_n' J_{-n} - J_n J_{-n}'$

$J_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} (x/2)^{n+2r}$

$J_n = \frac{x^n}{2^n \Gamma(n+1)}$ ——— "minus" ———; $J_{-n} = \frac{x^{-n}}{2^{-n}\Gamma(-n+1)}$

$J_n' = \frac{n\,x^{n-1}}{2^n \Gamma(n+1)}$ ——— ———; $J_{-n}' = \frac{-n\,x^{-n-1}}{2^{-n}\Gamma(-n+1)}$

$\Delta(J_n, J_{-n}) = \frac{1}{x}\left[\frac{n}{\Gamma(n+1)\Gamma(1-n)} + \frac{n}{\Gamma(n+1)\Gamma(1-n)}\right] =$

$\frac{2}{x}(\sin n\pi)/\pi$

i.e $\Delta = 0$ only if n is an integer or zero.

* Therefore J_n and J_{-n} are dependent solutions only if $n \in \mathbb{Z}$, otherwise the two solutions are independent. i.e when $n \in \mathbb{R} - \{\mathbb{Z}\}$ and hence $\Delta \neq 0$

when $\Delta = 0$ i.e $\Delta(J_n, J_{-n}) = \frac{2\sin\pi}{\pi x} = 0$.

$\therefore \sin\pi = 0$ $n = 0$, or n an integer

Thus $J_n(x)$, $J_{-n}(x)$ are two independent solutions if $n \neq$ an integer

G.S $y_{G.S} = A J_n(x) + B J_{-n}(x)$. $\leftarrow (n \neq 0,\ n \neq \text{integer})$

If $n = 0$ or $n =$ integer $\exists$ only 1 solution and we get another soln

$$Y_n(x) = \frac{(J_n(x)\cdot\cos n\pi) - J_{-n}(x)}{\sin n\pi}$$

Notice: ① Y_n is not defined when n is an integer , $\cos n\pi = (-)^n$

$\Rightarrow$ if $Y_n = \frac{0}{0}$. The other solution when n integer

$$\lim_{\epsilon\to 0} Y_{n+\epsilon}(x) = \frac{Y_{n+\epsilon}(x)\cos(n+\epsilon)\pi - J_{-(n+\epsilon)}(x)}{\sin(n+\epsilon)\pi} = \lim_{\epsilon\to 0} Y_{n+\epsilon}(x)$$

Bessel's Definite Integral.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta$$

we have

$$e^{ix\sin\theta} = \sum_{-\infty}^{\infty} e^{in\theta} J_{-n}(x) = \sum_{m=-n}^{\infty} e^{im\theta} J_m(x) + e^{in\theta} J_n(x)$$

Multiplying both sides by $e^{in\theta}$ and integrating from $-\pi$ to π

$$\int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}\, d\theta = \sum_{m\neq n} J_m(x) \int_{-\pi}^{\pi} e^{i(m-n)\theta}\, d\theta + J_n(x)\int_{-\pi}^{\pi} d\theta$$

$$= \sum_{m\neq n} J_m(x) \left| \frac{e^{i(m-n)\theta}}{i(m-n)} \right|_{-\pi}^{+\pi} + 2\pi J_n(x)$$

$$\int_{-\pi}^{\pi} \underset{\text{even}}{\cos}(x\sin\theta - n\theta)\, d\theta + i\int_{-\pi}^{\pi} \underset{\text{odd}}{\sin}(x\sin\theta - n\theta)\, d\theta = 0 + 2\pi J_n(x)$$

$$2\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta)\, d\theta + 0 = \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & f(x) \text{ even} \\ 0 & f(x) \text{ odd} \end{cases}$$

$$2\pi J_n(x)$$

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta)\, d\theta = J_n(x)$$

Remains to show that $\left| \frac{e^{i(m-n)\theta}}{i(m-n)} \right|_{-\pi}^{+\pi} = \frac{1}{i(m-n)}\left[e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right]$

$$= \frac{1}{i(m-n)} \times 2i\sin(m-n)\pi = 0 \qquad (n, m \text{ integers})$$

[2] $x =$ ——— [3] taking the derivative of the original relation w.r.t. θ then putting $\theta = 0$ we obtain [2]

[3]

[4] derivating the original relation twice then putting $\theta = \frac{\pi}{2}$.

1. Normal Form:

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \quad \text{---(1)}$$

To simplify → derive a more simple form i.e. instead of 2 term of (1,2) wanted one only to remain i.e. 1st derivative vanishes

Let $y = uv \Rightarrow y' = uv' + vu'$; $y'' =$ } ---(2)

$$y'' = uv'' + 2u'v' + vu''$$

$$uv'' + 2u'v' + vu'' + P(uv' + vu') + Q(uv) = R$$

$$\Rightarrow u\,v'' + (2u' + Pu)\,v' + (u'' + Pu' + Qu)\,v = R \quad \text{---(3)}$$

$$2u' + Pu = 0 \quad \text{i.e} \quad 2\frac{du}{u} = -P\,dx$$

$$2\ln u = -\int \frac{P}{2}\,dx \Rightarrow u = e^{-\frac{1}{2}\int P\,dx}$$

$$y = uv = e^{-\frac{1}{2}\int P\,dx}$$

$$\text{i.e} \quad uv'' + (u'' + Pu' + Qu)\,v = R \quad \text{---(3)}$$

Which is called [the normal form of eqtn (1)]

example: Reduce to the normal form, then Solve the eqtn

$$y'' - (2/x)\,y' + (1 + \tfrac{2}{x^2})\,y = x\sin 3x \quad (1)$$

Coeffic of y' is $P = -\frac{2}{x} \Rightarrow$

$$u = e^{+\int (2dx/x)} \Rightarrow u = e^{2\ln x} = x^2 = u$$

$$y = uv = xv \Rightarrow y' = xv' + vx \Rightarrow y'' = xv'' + 2v' \quad (2)$$

$$x\,v'' + 2v' - \frac{2}{x}(xv' + v) + (1 + \tfrac{2}{x^2})\,xv = x\sin 3x$$

$$xv'' - \frac{2}{x}v + xv + \frac{2}{x}v = xv'' + xv' = x\sin 3x$$

$$\therefore \quad v'' + v = \sin 3x \quad \text{[which is the normal form]}$$

$(D^2+1)v = \sin 3x$

which is a linear eqtn with const Coeff.

A.E $= \lambda^2+1=0 \Rightarrow \lambda = \pm i$

$\therefore v_{cf} = A\cos x + B\sin x$

$P.I = \frac{1}{D^2+1}\sin 3x = \frac{\sin 3x}{-9+1} = -\frac{1}{8}\sin 3x$

$\therefore y = xv = x\left[(A\cos x + B\sin x) - \frac{\sin 3x}{8}\right]$.

* Solve: $y'' - \frac{2}{x}y' + \left(1+\frac{2}{x^2}\right)y = xe^x$.

$P = -\frac{2}{x} \Rightarrow y = \left(e^{-\frac{1}{2}\int -\frac{2}{x}dx}\right) x(v) = e^{\ln x} v = xv$

$y' = v + xv'$, $y'' = xv'' + v' + v' = xv'' + 2v' \Rightarrow$

$xv'' + 2v' - \frac{2}{x}(v + xv') + \left(1+\frac{2}{x^2}\right)xv = xe^x$

notice the Coefficient of (v') must vanish.

$\mapsto$ i.e $xv'' + \left(x + \frac{2}{x} - \frac{2}{x}\right)v = xe^x$

$xv'' + xv = xe^x \mapsto v'' + v = e^x$ (Normal Form)

$\mapsto (D^2+1)v = e^x$ i.e eqtn with Const Coeffs

A.E: $\lambda^2+1=0$, $\lambda = \pm i$

$v_{cf} = A\cos x + B\sin x$

* $v_{P.I} = \frac{1}{D^2+1}e^x = \frac{1}{2}e^x$

$v = A\cos x + B\sin x + \frac{1}{2}e^x$, $y = xv$

$\Rightarrow y = Ax\cos x + Bx\sin x + \frac{1}{2}xe^x$

* Put in the normal form, hence Solve

$y'' - 4xy' + (4x^2-1)y = -3e^{x^2}\sin 2x$.

* Solve: $x^2(x-1)y'' - x(2+4x+x^2)y' + (2+4x+x^2)y = -x^4 - 2x^3$

Legendre's Polynomials.

$(1+tx)^{-1} = \sum_{n=0}^{\infty} (-tx)^n$

$(1+t)^{-1/2} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}(-t)^n$

Consider the Legendre DE

$$[(1-x^2)D^2 - 2xD + n(n+1)]y = 0$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad [1]$$

which is a 2nd order linear D.E, depending on the parameter (n).

Its solution depends on (n), and can be found using the power series method

$y = \sum_{r=0}^{\infty} a_r x^{c+r}$ finding Dy & D^2y and substituting in [1],

then equating the coefficient of the $x^{(t)}$ to (0).

and this done by equating the coefficient of the least power of x to zero

therefore we obtain

$c(c-1) = 0$ "which is the indicial eqt'n"

which gives 2-values for (c), repeating the same procedure we obtain the solutions of [1] in a series form. When $n \in \mathbb{Z}^+$; the case which concern us now, the solution is a Polynomial.

A Polynomial of x of degree(n) which is denoted by $P_n(x)$

$P_n(x)$ and $n \in \mathbb{Z}^+$. $\quad P_n(x) = \frac{1}{2^n}\sum_{r=0}^{m} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r)!} x^{n-2r}$

$$P_n(x) = \sum_{r=0}^{m} [(-)^r x^{(n-2r)} (2n-2r)! / (2^n r!(n-r)!(n-2r)!)]$$

$$P_n(x) = \sum_{r=0}^{m} \left(\frac{(-)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} \right) x^{n-2r} \qquad (1)$$

Const. out of $\sum$

where $m = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$

* Rodrigues formulae

Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof: $\frac{d^n}{dx^n}(x^{2n-2r}) = [(2n-2r)(2n-2r-1)\cdots(n-2r+1)]x^{n-2r}$

=> to get the factorial form =>

① $(2n-2r-(n-1))\, D^{n-(n-1+1)}\, x^{2n-2r-(n-1+1)}$

$$\frac{d^n}{dx^n}(x^{2n-2r}) = \frac{(2n-2r)!}{(n-2r)!}x^{n-2r}$$

$n-2r \geq 0$

$\frac{n}{2} \geq r \in \mathbb{Z}^+$ (2)

① $D^n(x^{2n-2r}) = (2n-2r)(2n-2r-1)\, D^{n-2} x^{2n-2r-2}$

Substituting from (2) in (1) ⇒

$$P_n(x) = \frac{1}{2^n} \sum_{r=0}^{m} \frac{(-)^r}{r!(n-r)!} \frac{d^n}{dx^n}\left(x^{2n-2r}\right) =$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[\sum_{r=0}^{m} \frac{(-)^r\, n!}{r!(n-r)!} (x^2)^{n-r}\right] \Rightarrow$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[\sum_{r=0}^{m} {}^nC_r\, (-)^r (x^2)^{n-r}\right] =$$

But $\sum_{r=0}^{m} {}^nC_r\, (-)^r (x^2)^{n-r} = (x^2-1)^n$

i.e. Its binomial expansion ⇒

Notice: Coefficients of (-ve) powers of x vanish, hence $\sum_{r=0}^{m}$.

$$P_n(x) = \frac{1}{2^n n!}\left[\frac{d^n}{dx^n}(x^2-1)^n\right]$$

*

Generating function for $P_n(x)$

Expanding the h^n below in terms of powers of h, in Taylor's series, the coefficients of h^n shows $P_n(x)$ which is the Legendre polynomial of the n rank.

$$(1-2xh+h^2)^{-1/2}$$

$$[1+(\pm\lambda)]^{-1} = \sum_{n=0}^{\infty} (\pm\lambda)^n$$

$$[1+(\pm\lambda)]^{-1/2} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} (\pm\lambda)^n$$

which depends on (x) and (h), of course powers of (h)

$$(1-\lambda)^{-1/2} = 1 + \frac{1}{2}\lambda + \frac{1\cdot 3}{2\cdot 4}\lambda^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\lambda^3 + \cdots$$

$$+ \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\lambda^n + \cdots$$

Therefore, $(1-2xh+h^2)^{-1/2} = (1-(2xh-h^2))^{-1/2}$

where $\lambda = (2xh-h^2)$ ⇒

$$(1-(2xh-h^2))^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, h^n$$

$$(1-(2xh-h^2))^{-1/2} = 1 + \frac{1}{2}(2xh-h^2) + \frac{1\cdot 3}{2\cdot 4}(2xh-h^2)^2 +$$

$$\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(2xh-h^2)^3 + \cdots$$

$$(1-(2xh-h^2))^{-\frac{1}{2}} = 1 + xh + (\tfrac{3}{2}x^2 - 1/2)h^2 + (\quad)h^3 + \cdots \quad (1)$$

where $P_0(x) = 1$ a zero degree polynomial.

$P_1(x) = x$ a 1st degree polynomial

" a 2nd degree polynomial " $P_2(x) = (\frac{3}{2}x^2 - \frac{1}{2})$

* Find $P_n(1)$, $P_n(-1)$

- Put $x = 1$ in (1) inorder to find $P_n(1)$

$$(1-2h+h^2)^{-\frac{1}{2}} = \sum_0^\infty P_n(1)h^n = (1-h)^{-1}$$

where $(1-2h+h^2) = (1-h)^2$

$$\Rightarrow (1-h)^{-1} = \sum_0^\infty P_n(1)h^n = 1+h+h^2+\cdots$$

which is a geometric progression.

$$\boxed{P_n(1) = 1} \quad (I)$$

* Put $x = -1$ in (1)

$$(1+2h+h^2)^{-\frac{1}{2}} = (1+h)^{-1} = \sum_{n=0}^\infty P_n(-1)h^n =$$
$$1-h+h^2-h^3+\cdots + (-1)^n h^n$$

Therefore, $\boxed{P_n(-1) = (-1)^n}$ (II)

Find $P_n(\cos\theta) = P_n(x)$ use the generating h^n expand it

$$2\cos\theta = e^{i\theta} + e^{-i\theta} \quad ; \quad h^2 \times 1 = h^2 e^{i\theta} e^{-i\theta}$$

$$\Rightarrow (1-he^{i\theta})^{-1/2} \cdot (1-he^{-i\theta})^{-1/2}$$

Recurrence Relations.

* $n P_n(x) - x(2n-1) P_{n-1} + (n-1) P_{n-2} = 0$ (1)

Let $V = V(x,h) = (1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) h^n$ (3)*

$$\frac{\partial V}{\partial h} = -\frac{1}{2}(1 - 2xh + h^2)^{-\frac{3}{2}} \cdot (-2x + 2h)$$

$$(1 - 2xh + h^2) \frac{\partial V}{\partial h} = (x - h)(1 - 2xh + h^2)^{-\frac{1}{2}} \quad (4)$$

diff the R.H.S of (3) w.r to h ⇒

$$(1 - 2xh + h^2) \frac{\partial V}{\partial h} = (1 - 2xh + h^2) \sum_{0}^{\infty} n P_n(x) h^{n-1} \Rightarrow$$

$$(x - h) \sum_{0}^{\infty} P_n(x) h^n$$

equating the Coefficients of Powers of (h) in both sides $\frac{\partial V}{\partial h}$ ⇒

$$\text{L.H.S} = n P_n - 2x(n-1) P_{n-1} + (n-2) P_{n-2} = x P_{n-1} - P_{n-2} = \text{R.H.S}$$

$$n P_n - x(2n-1) P_{n-1} + (n-1) P_{n-2} = 0$$

---

* $x P_n' - P_{n-1}' = n P_n$ (II)

$\frac{\partial V}{\partial x}$ ⇒ Comparing Coeff. of h^n (x) (3)* (II)

Proof: $(1 - 2xh + h^2) \frac{\partial V}{\partial x} = -\frac{1}{2}(1 - 2xh + h^2)^{-\frac{3}{2}} (-2h)$ 5

$$= (1 - 2xh + h^2)^{-\frac{1}{2}} h = (1 - 2xh + h^2) \frac{\partial V}{\partial x} \quad (5)$$

(4), (5) ⇒ $(x - h) \frac{\partial V}{\partial x} = h \frac{\partial V}{\partial h}$ ∴

∴ diff (3)* w.r to x ⇒

$$(x - h) \sum_{0}^{\infty} P_n' h^n = h \sum_{0}^{\infty} n P_n h^{n-1}$$

Comparing the Coefficient of h^n in both sides ⇒

$$x P_n' - P_{n-1}' = n P_n \quad (2)$$

* $P_n' - x P_{n-1}' = n P_{n-1}$ (III)

Proof:

From (I) we can write it as

$n P_n - x(n+(n-1))P_{n-1} + (n-1) P_{n-2} = 0$

i.e. $n(P_n - x P_{n-1}) - (n-1)(x P_{n-1} - P_{n-2}) = 0$

diff w.r to x, ⇒

$n(P_n' - x P_{n-1}' - P_{n-1}) - (n-1)(x P_{n-1}' + P_{n-1} - P_{n-2}') = 0$

$n(P_n' - x P_{n-1}') - (n-1)(x P_{n-1}' - P_{n-2}') = (n+n-1) P_{n-1}$

(Look above (reason) ↩) (How to get or relate it to something we know)

From (II) Putting (n-1) instead of (n) ⇒

$n(P_n' - x P_{n-1}') - (n-1)x((n-1) P_{n-1}) = (2n-1) P_{n-1}$

$\therefore n(P_n' - x P_{n-1}') = ((n-1)^2 + 2n-1) P_{n-1}$

$n(P_n' - x P_{n-1}') = n^2 P_{n-1}$

Finally we get $(P_n' - x P_{n-1}') = n P_{n-1}$ (III)

* $P_{n+1}' - P_{n-1}' = (2n+1) P_n$ (IV)

Proof: deduce from III, II

From III Putting (n+1) instead of (n),

$P_{n+1}' - x P_n' = (n+1) P_n$ ⑥

(II) + ⑥ ⇒

$P_{n+1}' - P_{n-1}' = (2n+1) P_n$

* Show that: $P_{2n+1}(0) = 0 ; P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$

* $x P_n' - P_{n-1}' = n P_n$ (II) ; $P_n' - x P_{n-1}' = n P_{n-1}$ (III)

V. $x \times$ (III) $-$ (II) $\Rightarrow$

$$(1-x^2) P_{n-1}' = n x P_{n-1} - n P_n \quad \text{V.}$$

* Laplace Integral for $P_n(x)$

* $$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2-1} \cdot \cos\theta \right)^n d\theta$$

Since $$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}} \qquad (a > b) \quad (1)$$

$$\int_0^{\pi} \frac{d\theta}{a\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) + b\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)} =$$

$$\int_0^{\pi} \frac{d\theta}{(a+b)\cos^2\frac{\theta}{2} + (a-b)\sin^2\frac{\theta}{2}} = \int_0^{\pi} \frac{\sec^2\frac{\theta}{2}\, d\theta}{(a+b) + (a-b)\tan^2\frac{\theta}{2}} =$$

$$\frac{2}{(a-b)} \int_{\tan\frac{\theta}{2}=0}^{\tan\frac{\theta}{2}=\infty} \frac{d\tan(\theta/2)}{\left(\frac{a+b}{a-b}\right) + \tan^2\left(\frac{\theta}{2}\right)} = \frac{2}{a-b} \cdot \frac{1}{\sqrt{\frac{a+b}{a-b}}} \left[\tan^{-1}\left(\tan\frac{\theta}{2} \sqrt{\frac{a+b}{a-b}} \right) \right]_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}} .$$

* In (1), Put $a = 1 - hx$, $b = -h\sqrt{x^2-1}$

$$a^2 - b^2 = 1 - 2hx + h^2x^2 - h^2(x^2-1) = 1 - 2ht + h^2 \quad (2)$$

Substi from (2) in (1)

$\Rightarrow$ (1) becomes

$$R.H.S = \frac{\pi}{\sqrt{1-2hx+h^2}} = \int_{\theta=0}^{\theta=\pi} \frac{d\theta}{1-hx-h\sqrt{x^2-1}.\cos\theta} = L.H.S$$

$$\Rightarrow \pi \sum_r P_n(x).h^n = \int_0^\pi \left[1-h(x+\sqrt{x^2-1}.\cos\theta)\right]^{-1} d\theta$$

$$= \int_0^\pi \sum_{n=0}^{\infty} \left(x+\sqrt{x^2-1}.\cos\theta\right)^n h^n d\theta =$$

$$\sum_{n=0}^{\infty} \int_0^\pi \left(x+\sqrt{x^2-1}\cos\theta\right)^n h^n \, d\theta$$

Comparing the two sides we obtain

$$\pi P_n(x) = \int_0^\pi \left(x+\sqrt{x^2-1}\cos\theta\right)^n d\theta$$

$$\int_{-1}^{+1} P_m(x) P_n(x) \, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

Using Rodrigues Formulas

$$\int_{-1}^{+1} P_m(x) P_n(x) \, dx = \frac{1}{2^{m+n} m!.n!} \int_{-1}^{+1} \underbrace{\frac{d^m}{dx^m}(x^2-1)^m}_{u} \cdot \underbrace{\frac{d}{dx^n}(x^2-1)^n dx}_{dv}$$

$$\int u\,dv = uv - \int v\,du$$

$$\frac{1}{2^{m+n} m!.n!} \left\{ \left[\frac{d^m}{dx^m}(x^2-1)^m \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \right]_{-1}^{+1} - \int_{-1}^{+1} \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \, dx \right.$$

بتطبيق نظرية ليبنز لايجاد $\frac{d^n}{dx^n}$ يحتوي على قوى متتالية لكل من $(x-1)$ وقوى
أعلى للقوى $(x+1)$ ، وذلك لأن رتبة المشتقة $(n-1)$
، بينما تبدأ قوى x من الصفر إلى n

$$\int_{-1}^{+1} P_m(x) \cdot P_n(x)\, dx = \frac{1}{2^{m+n} m!\, n!} \int_{-1}^{+1} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^m \cdot \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n\, dx$$

وبتكرار التكامل (n مرات)

$$\int_{-1}^{+1} P_m(x) \cdot P_n(x)\, dx = \frac{(-1)^n}{2^{m+n} m!\, n!} \int_{-1}^{+1} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^m \cdot (x^2-1)^n\, dx$$

* Prove that: $\int_{-1}^{+1} x^k P_n(x)\, dx = 0 \quad k = 0, 1, 2, \ldots n-1$

* Prove that: $P_n(\cos\theta) = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} [2\cos n\theta + \frac{1 \cdot 2n}{2 \cdot (2n-1)} 2\cos(n-2)\theta + \frac{1 \cdot 3 \cdots 2n(2n-2)}{2 \cdot 4 \cdots (2n-1)2n-3} \cos$

D.E's Variation of Parameters

The method will be illustrated by the following example :

$$(D^2+4)y = 4\tan 2x \qquad (1)$$

A.E. $\lambda^2+4=0 \Rightarrow \lambda = \pm 2i$

$y_{C.F} = A\cos 2x + B\sin 2x$, A, B are two arbitrary Constants

$$y = u\cos 2x + v\sin 2x \qquad (\text{H})$$

where u, v are two functions of x

$$Dy = u'\cos 2x + v'\sin 2x - 2u\sin 2x + 2v\cos 2x$$

نختار u & v بحيث

$$u'\cos 2x + v'\sin 2x = 0 \qquad (2)$$

$$Dy = -2u\sin 2x + 2v\cos 2x \qquad (3)$$

$$D^2y = -4u\cos 2x - 2u'\sin 2x - 4v\sin 2x + 2v'\cos 2x \qquad (4)$$

Subst. from (4) in (1)

$$(4) + 4(\text{H}) = -2u'\sin 2x + 2v'\cos 2x = 4\tan 2x \qquad (5)$$

نحل (5) & (2) معاً

$$u'\cos 2x + v'\sin 2x = 0 \qquad \times(2\sin 2x)$$
$$-2u'\sin 2x + 2v'\cos 2x = 4\tan 2x \qquad \times(\cos 2x) \qquad (+)$$

$$2v'\sin^2 2x + 2v'\cos^2 2x = 4\sin 2x = 2v' \Rightarrow$$

$$v' = 2\sin 2x \Rightarrow$$

$$u' = -\frac{2\sin^2 2x}{\cos 2x} = -2\tan 2x \cdot \sin 2x$$

* $v = 2 \cdot 2\int -2 \sin 2x\, dx = -\cos 2x = v$ (i)

* $u = \int -2 \tan 2x \cdot \sin 2x\, dx = \int \frac{-2\sin^2 2x}{\cos 2x}\, dx$

$$\int -2\left(\frac{1-\cos^2 2x}{\cos 2x}\right) dx$$

$$\int -2(\sec 2x - \cos 2x)\, dx$$

$$-\ln|\sec 2x + \tan 2x| + 2\int \cos 2x\, dx$$

$$-\ln(\sec 2x + \tan 2x) + \sin 2x + A \quad \text{(ii)}$$

Subst. in (2), (2)

$$y = A\cos x + \sin 2x - \ln(\sec 2x + \tan 2x) + B\sin x - \sin x \cos 2x$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{(-1)^n}{2^{m+n} m!\, n!} \int_{-1}^{1} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^m \cdot (x^2-1)^n\, dx$$

1st If $m \neq n$; we suppose $n > m$ without specillity

without loss of generality, and this is ~~because~~ m, n in the integrand.

$n > m$ is alike putting n instead of m

$\Rightarrow$ the order of integration $> 2m$, for $n > m$.

and the order of integration $> 2m$.

* order of integration $> 2m$. the heighest power of the expansion $(x^2-1)^m$ which is x^{2m}

* If $n = m$ then (1) becomes

$$\int_{-1}^{+1} P_m \cdot P_n\, dx = 0 \quad (2)$$

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \times \left\{\frac{d^{2n}}{dx^{2n}} (x^2-1)^n\right\} dx$$

$$= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \cdot (2n)!\, dx = \frac{(2n)!}{2^{2n}\cdot(n!)} \int_0^1 (1-x^2)^n\, dx$$

$$= \frac{2\,(2n)!}{2^{2n}(n!)^2} \int_0^{\pi/2} \cos^{2n+1} d\theta .$$

Putting $x = \sin\theta \Rightarrow$

$$= \frac{2\,(2n)!}{2^{2n}(n!)^2} \times \int_0^{\pi} \cos^{2n+1}\theta\, d\theta = \frac{2\cdot(2n)!}{2^{2n}(n!)^2} \times \frac{2n(2n-1)\cdots 4\times 2}{(2n+1)(2n+1)\cdots 3\times 1} =$$

$$\frac{2\,(2n)!}{2^{2n}(n!)^2} \cdot \frac{[2n(2n-2)\cdots 4\times 2]^2}{(2n+1)!} = \frac{2\,(2n)!}{2^{2n}(n!)^2} \times \frac{[2^n n!]^2}{(2n+1)!} = \frac{2}{2n+1}$$

Prove that:

$$P_n(\cos\theta) = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \left[2\cos n\theta + \frac{1^2 n}{2(2n-1)} 2\cos(n-2)\theta \right.$$

$$\left. + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n+1)(2n-3)} 2\cos(n-4)\theta + \cdots \right.$$

Since:

$$(1 - 2h\cos\theta + h^2)^{-1/2} = \sum_0^{\infty} P_n(\cos\theta)\, h^n =$$

$$\left[1 - h\left(\frac{e^{i\theta} + e^{-i\theta}}{\cdot}\right) + h^2\, e^{i\theta} \cdot e^{-i\theta} \right]^{-1/2} =$$

$$(1 - he^{i\theta})^{-1/2} \cdot (1 + he^{-i\theta})^{-1/2}$$

$$(1-\lambda)^{-1/2} = 1 + \frac{1}{2}\lambda + \frac{1\cdot 3}{2\cdot 4}\lambda^2 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\lambda^n + \cdots$$

$$= \left[1 + \frac{1}{2} h e^{i\theta} + \frac{1\cdot 3}{2\cdot 4} h^2 e^{2i\theta} + \cdots + \frac{1\cdot 3\cdot(2n-3)}{2\cdot 4\cdots(2n-2)} h^{n-1} e^{(n-1)i\theta} \right.$$

$$\left. + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} h^n e^{ni\theta} + \cdots \right] \times$$

$$\left[1 + \frac{1}{2} h e^{-i\theta} + \cdots + \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)} h^{n-1} e^{-(n-1)i\theta} + \right.$$

$$+\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}h^n e^{-ni\theta}\Big]$$

Comparing coefficients of h^n in the two sides $\Rightarrow$

$$P_n(\cos\theta)=\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\left[e^{ni\theta}+e^{-ni\theta}\right]+\frac{1}{2}\cdot\frac{2n}{2n-1}\left(e^{i(n-2)\theta}+e^{-i(n-2)\theta}\right)$$

$$+\frac{1\cdot3}{2\cdot4}\cdot\frac{2n(2n-2)}{(2n-1)(2n-3)}\left(e^{i(n-4)\theta}+e^{-i(n-4)\theta}\right)+\cdots$$

* Solve:

$$\int_{-1}^{+1}x^k P_n(x)\,dx=0 \qquad k=0,1,2,\cdots n-1$$ (orthogonality property of Legendre polynomials)

Rodrigues formulas for $P_n(x)$ is utilized

$$\int_{-1}^{+1}x^k P_n(x)\,dx=\frac{1}{2^n n!}\int_{-1}^{+1}x^k\cdot\frac{d}{dx^n}(x^2-1)^n\,dx=$$

$u \qquad dv$

by integration by parts $\Rightarrow$

$$=\frac{1}{2^n n!}\left[x^k\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{+1}-\frac{k}{2^n n!}\int_{-1}^{+1}x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$=(-)^2\frac{k(k-1)}{2^n n!}\int_{-1}^{+1}x^{k-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n$$

وبتكرار هذه العملية (التكامل بالتجزئ) k من المرات

$$=(-)^k\frac{k(k-1)\cdots1}{2^n\cdot n!}\int_{-1}^{+1}x^{k-k}\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n=$$

$$(-)^k\frac{k!}{2^n\cdot n!}\int_{-1}^{+1}\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n=$$

$$= \frac{(-)^K K!}{2^n \cdot n!} \left[\frac{d^{n-K-1}}{dx^{n-K-1}} (x^2-1)^n \right]^{+1}_{-1} = 0$$

One integral belonging to the Complementary function.

Solve the DE:

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = (2x-1)x^2 \quad (1)$$

By inspection we can have $y_{C.F} = x$ as a solution.

To find the general solution we suppose that $y = xv$?

and v is wanted to be found.

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad ; \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} \quad (2)$$

Subst. from (2) in (1) $\Rightarrow$

in such a kind of DE's (Which must always vanish)

$$x^3\frac{d^2v}{dx^2} + 2x^2\frac{dv}{dx} - 3x^2\frac{dv}{dx} - 3xv + 3xv = (2x-1)x^2$$

$$x^3\frac{d^2v}{dx^2} - x^2\frac{dv}{dx} = x^2(2x-1)$$

$$x\frac{d^2v}{dx^2} - \frac{dv}{dx} = (2x-1)$$

Let $\frac{dv}{dx} = z \Rightarrow$ and dividing by x ;

$$\frac{dz}{dx} - z = (2 - 1/x) \quad ; \quad e^{\int \frac{1}{x}dx} = \frac{1}{x}$$

$$\frac{dz}{z} - z\,dx = 2\,dx - \frac{dx}{x}$$

$$z \times \frac{1}{x} = \int (2 - \frac{1}{x}) \cdot \frac{1}{x}\,dx$$

Solve:

$x^2(x-1)\,y'' - x(2+4x+x^2) + (2+4x+x^2)y = x^4 - 2x^3$

"$y = x$ is a solution"

* $2\left[\frac{n}{x} J_n(x)\right] = J_{n-1}(x) + J_{n+1}(x)$ (3) for $n \geq 1/2$

$\frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x) \Rightarrow J_{3/2} = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) =$

even powers even power "symmetry in Mathematics"

$\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right] = J_{3/2}$.

* $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ (4)

$2 J_{-1/2}'(x) = J_{-3/2}(x) - J_{1/2}(x)$

$J_{-3/2}(x) = 2 J_{-1/2}'(x) + J_{1/2} = 2 \frac{d}{dx}\left[\sqrt{\frac{2}{\pi x}} \cos x\right] + \sqrt{\frac{2}{\pi x}} \sin x$.

and in the same way we can find $J_{3/2}$, $J_{5/2}$.

* Using the part of Bessel fns numbered "9th" to obtain $J_{3/2}$, $J_{5/2}$.

* Examination Special fns 2 questions Bessel, Legendre

3 questions in DE's

DE's of 1st order ; Linear DE with Const Coefficients,

Homogenous DE's , Simultaneous , miscellaneous

* prove that $\int_{-1}^{+1} x P_{n-1}(x) P_n(x)\, dx = \frac{2n}{4n^2-1}$

recurrence relation. is used

* Prove that $\int_{-1}^{+1} x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

~~use~~ then put $\boxed{n-1}$

apply for QM ("a particle") Compare with $[G_2(G_1)](0 G_1) = G_1$

Chaos is characterized by: wanted a particular $x = x_0$ to Spread in the whole x space to return to x_0

the Cause = (hidden reason) for the Spread is to Counteract the (rongdings separation) in the path ie evolution.

* Use variation of Parameters to Solve

$(D^2 + 1) y =$

Hermite polynomials in terms of their generating fn

$$e^{2xt} \cdot e^{-t^2} = \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \quad , \quad H_n(x) = \sum_{n=0}^{[n/2]} \frac{(-1)^n \, n! \, (2x)^{n-2k}}{k! \, (n-2k)!}$$

(given a Polynomial sets search for their generating functions)

Physics and mathematics, the 1st is received by the eyes and the 2nd is received by the head

Legendre's Recurrence Relations

Prove: $\boxed{n P_n - x(2n-1) P_{n-1} + (n-1) P_{n-2} = 0}$ (I)

Let $V(x,h) = V = (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n h^n \Rightarrow$

$\frac{\partial V}{\partial h} = -\frac{1}{2}(1-2xh+h^2)^{-1}(1-2xh+h^2)^{-1/2} \times -2(x-h) = \sum_{n=0}^{\infty} n P_n h^{n-1}$

$\Rightarrow \frac{1}{V^2}\frac{\partial V}{\partial h} = (x-h)V = \frac{1}{V^2}\sum_{n=0}^{\infty} n P_n h^{n-1}$ (1)

$V^2 = (1-2xh+h^2)^{-1} \Rightarrow$

$(x-h)\sum_{n=0}^{\infty} P_n h^n = (1-2xh+h^2)\sum_{n=0}^{\infty} n P_n h^{n-1} \Rightarrow$

$\sum_{n=0}^{\infty}(n P_n)h^{n-1} + \sum_{n=0}^{\infty}(-2xn P_n)h^n + \sum_{n=0}^{\infty}(n P_n)h^{n+1} = \sum_{n=0}^{\infty}(x P_n)h^n + \sum_{n=0}^{\infty}(-P_n)h^{n+1}$

(1) (2) (3) (2) (3)

$\sum_{n=0}^{\infty} n P_n h^{n-1} + \sum_{n=0}^{\infty} -x(2n+1) P_n h^n + \sum_{n=0}^{\infty}(n+1) P_n h^{n+1} = 0$

$0 + P_1 + 2P_2 h + \sum_{n=3}^{\infty} n P_n h^{n-1}$

$+ (-x P_0) + (-x \times 3 P_1)h + \sum_{n=2}^{\infty} -x(2n+1) P_n h^n$

$+ (-P_0)h + \sum_{n=1}^{\infty}(n-1) P_n h^{n+1} = 0$

$(-xP_0 + P_1) + (2P_2 - 3xP_1 - P_0)h + \sum_{n=3}^{\infty} n P_n h^{n-1} + \sum_{n=2}^{\infty} -x(2n+1) P_n h^n + \sum_{n=1}^{\infty}(n-1) P_n h^{n+1} = 0$

$(=0)$ $(=0)$ (1) (2) (3)

* in (1)(2)(3) let the summation starts from $n=3 \Rightarrow$

(2) $n=2 \mapsto n=3$ and $(2n+1) \mapsto 2(n-1)+1 = 2n-1$, $P_n \mapsto P_{n-1}$

(3) $n=1 \mapsto n=3$ and $(n+1) \mapsto (n-2)+1 = n-1$, $P_n \mapsto P_{n-2}$

$\Rightarrow \cancel{\quad}\, h^0 + \cancel{\quad}\, h + \sum_{n=3}^{\infty}[n P_n - x(2n-1) P_{n-1} + (n-1) P_{n-2}] h^{n-1} = 0$

Since $\{h^n\}$, $n = 0, 1, 2, \ldots$ are linearly independent then the coefficient of powers of h are equivalent to zero $\Rightarrow$

$n P_n - x(2n-1) P_{n-1} + (n-1) P_{n-2} = 0$ ∎

(64)

Prove: $\boxed{x P_n' - P_{n-1}' = n P_n}$

$$V(x,h) = (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n h^n$$

$$\frac{\partial V}{\partial x} = -\frac{1}{2} V^2 \left(\sum_{n=0}^{\infty} P_n h^n\right)(-2h) = h \sum_{n=0}^{\infty} P_n h^n \; (1-2xh+h^2)^{-1}$$

$$(1-2xh+h^2)\frac{\partial V}{\partial x} = h \sum_{n=0}^{\infty} P_n h^n = hV \quad (2)$$ Using (I.①) ⇒

$$\frac{1}{V^2}\frac{\partial V}{\partial h} = (1-2xh+h^2)\frac{\partial V}{\partial h} = (x-h)V = xV - hV \quad (1)$$

$$(1-2xh+h^2)\frac{\partial V}{\partial h} = \frac{1}{h}(1-2xh+h^2)\frac{\partial V}{\partial x}(x-h) \Rightarrow$$

$$(x-h)\frac{\partial V}{\partial x} - h\frac{\partial V}{\partial h} = \sum_{n=0}^{\infty} n P_n h^n = (x-h)\sum_{n=0}^{\infty} \frac{\partial P_n}{\partial x} h^n$$

$$\Rightarrow \cancel{n P_n = (x-h) P_n'} \Rightarrow \sum_{n=0}^{\infty} n P_n h^n = x \sum_{n=0}^{\infty} P_n' h^n + \sum_{n=0}^{\infty} -P_n' h^{n+1}$$

$$\sum_{n=0}^{\infty} -x P_n' h^n + \sum_{n=0}^{\infty} n P_n h^n = \sum_{n=0}^{\infty} -P_n' h^{n+1}$$

$$\sum_{n=0}^{\infty} (n P_n - x P_n') h^n = \sum_{n=0}^{\infty} -P_n' h^{n+1}$$

Starting the summation from h^{1-n}

$$\underbrace{(0 P_0 - x P_0')}_{(=0)} h^0 + \sum_{n=1}^{\infty} (n P_n - x P_n') h^n = \sum_{n=0}^{\infty} -P_n' h^{n+1}$$

$$= \sum_{n=1}^{\infty} -P_{n-1}' h^n$$

$$\Rightarrow n P_n - x P_n' = -P_{n-1}' \Rightarrow$$

$$x P_n' - P_{n-1}' = n P_n$$

(1) B

evaluate y' where $y = x^{x} = x^{[2]}$ or $x^{x^{x}} = x^{[3]}$ or $x^{x^{x^{x}}} = x^{[4]}$ or $x^{[n]}$ as $n \to \infty$ $\quad x^{[2]} = x^{x}$

$y = x^{x} \Rightarrow \ln y = x \ln x \Rightarrow \frac{y'}{y} = x \cdot \frac{1}{x} + \ln x = (1 + \ln x) \Rightarrow$

$D_x x^{[2]} = (1 + \ln x)\, x^{[2]} \Rightarrow$ if $y = x^{[2]} \Rightarrow \ln y = x^{[1]} \ln x \Rightarrow$

$\frac{y'}{y} = x^{[1]} \cdot \frac{1}{x} + \ln x \, D_x x^{[1]} \Rightarrow D_x x^{[2]} = \frac{1}{x} x^{[1]} x^{[2]} + \ln x \, x^{[1]} x^{[2]} + (\ln x)^2 x^{[1]} x^{[2]}$

$* \; y = x^{[3]} \Rightarrow \ln y = x^{[2]} \ln x \Rightarrow \frac{y'}{y} = x^{[2]} \cdot \frac{1}{x} + \ln x \cdot D_x x^{[2]} \Rightarrow$

$D_x x^{[3]} = \frac{1}{x} x^{[2]} x^{[3]} + \frac{1}{x} \ln x \, x^{[1]} x^{[2]} x^{[3]} + (\ln x)^2 x^{[1]} x^{[2]} x^{[3]}$

$* \; y = x^{[4]} \Rightarrow \ln y = x^{[3]} \ln x \Rightarrow \frac{y'}{y} = x^{[3]} \cdot \frac{1}{x} + \ln x \cdot D_x x^{[3]} \Rightarrow$

$D_x x^{[4]} = \frac{1}{x} (\ln x)^2 x^{[1]} x^{[2]} x^{[3]} x^{[4]} + (\ln x)^3 x^{[1]} x^{[2]} x^{[3]} x^{[4]} + (\ln x)^4 x^{[1]} x^{[2]} x^{[3]} x^{[4]} + \frac{1}{x} (\ln x)^2 x^{[1]} x^{[2]} x^{[3]} x^{[4]} + \frac{1}{x} x^{[3]} x^{[4]} \cdot$

$x^{[1]} = x \; ; \; x^{[2]} = x^{x} \; ; \; x^{[3]} = 1 \; ; \; x^{[4]} = x^{[1]} \cdot x^{[2]} \cdot \ldots \cdot x^{[n]} = x^{[n]}!$

$D_x x^{[n]} = \frac{1}{x} x^{[n-1]} x^{[n]} + \ln x \, D_x x^{[n-1]} x^{[n]}$

$D_x x^{[n]} = \left[\frac{1}{x} (\ln x)^{n-2} + (\ln x)^{n-1} + (\ln x)^{n} \right] x^{[1]} x^{[2]} \cdots x^{[n]} +$

$\frac{1}{x} (\ln x)^{n-3} x^{[2]} x^{[3]} \cdots x^{[n]} +$

$\frac{1}{x} (\ln x)^{n-4} x^{[3]} x^{[4]} \cdots x^{[n]} +$

$\frac{1}{x} (\ln x)^{n-5} x^{[4]} x^{[5]} \cdots x^{[n]} + \cdots +$

$\frac{1}{x} x^{[n-1]} x^{[n]} \cdot =$

$\left[\frac{1}{x} (\ln x)^{n-2} + (\ln x)^{n-1} + (\ln x)^{n} \right] x^{[n]}! + \frac{1}{x} \sum_{j=1}^{n-1} (\ln x)^{n-2-j} \frac{x^{[n]}!}{x^{[j]}!} =$

$(\ln x)^{n} x^{[n]}! + (\ln x)^{n-1} x^{[n]}! + \frac{1}{x} \sum_{j=1}^{n-1} (\ln x)^{n-1-j} \cdot x^{[n]}! / x^{[j]}!$

$y = x^{[3]} = x^{x^{x}} \Rightarrow \ln y = y_2 \ln x \; , \; \ln(\ln y) = \ln y_2 + \ln(\ln x)$

$y_2 \ln x$

(2) B

The Multiplication-theorem of Gauss and Legendre:

$$\varphi(x) = m^{mx}\left[\prod_{r=0}^{m-1}\Gamma\left(x+\frac{r}{m}\right)\right] / \Gamma(mx) = \sqrt{m}\cdot(2\pi)^{m-1},\ m \text{ is constant.}$$

1st By Euler Def. $\Gamma\left(x+\frac{r}{m}\right) = \lim_{n\to\infty}\left[(n-1)!\cdot n^{x+r/m}\right] / \prod_{k=0}^{n-1}(x+r/m+k)$

Also, $\Gamma(mx) = \lim_{mn\to\infty}\left[(mn-1)!\cdot(mn)^{mx}\right] / \prod_{k=0}^{mn-1}(mx+k)$ $\left[S_n = \frac{n}{2}(t_1+t_n)\right]$

* The numerator $\lim m^{mx}\left[\prod_{r=0}^{m-1}\left((n-1)!\cdot n^{x+r/m} / \prod_{k=0}^{n-1}(x+r/m+k)\right)\right] =$

$\lim_{n\to\infty}\left[\left(m^{mx}\cdot n^{mx}\cdot n^{\sum_{r=0}^{m-1} r/m}\right)\times\left((n-1)!\right)^m\right] / \left[\prod_{r=0}^{m-1}\prod_{k=0}^{n-1}(mx+r+km)\right]\times\left[m^{-mn}\right] =$

$\lim_{n\to\infty}\left[m^{mx}\cdot n^{mx}\cdot n^{m(m-1)/2m}\cdot m^{mn}\cdot\left[(n-1)!\right]^m\right] / \prod_{r=0}^{m-1}\prod_{k=0}^{n-1}(mx+r+km) \Rightarrow$

But, $\prod_{k=0}^{n-1}\left[\prod_{r=0}^{m-1}(mx+r+km)\right] =$

$\Gamma(mx)\,(mx+1)(mx+2)\cdots(mx+m-1)\cdot\Gamma(mx+0+m)(mx+1+m)(mx+2+m)\cdots(mx+2m-1)\cdot$

$\Gamma(mx+0+2m)(mx+2m+1)\cdots(mx+2m+m-1)\cdot\cdots$

$(mx+mn-m+0)(mx+mn-m+1)(mx+mn-m+2)\cdots(mx+mn-m+m-1) =$

$(mx)(mx+1)(mx+2)\cdots(mx+m-1)\cdot(mx+m)\cdots(mx+2m-1)\cdot$

$(mx+2m)\cdots(mx+2m-1)\cdot\cdots\cdot(mx+mn-m)\cdots(mx+mn-1) =$

$= \prod_{s=0}^{mn-1}(mx+s)\ . \Rightarrow$

$\lim_{n\to\infty}\left[(n-1)!^m\cdot m^{mx}\cdot n^{mx}\cdot m^{mn}\cdot n^{(m-1)/2} / \prod_{s=0}^{mn-1}(mx+s)\right] =$ numerator. ①

* denominator $= \Gamma(mx) = \lim_{mn\to\infty}\left[(mn-1)!\cdot m^{mx}\cdot n^{mx} / \prod_{s=0}^{mn-1}(mx+s)\right]$ ②

① / ② $= \lim_{mn\to\infty}\left[(n-1)!^m\cdot m^{mn}\cdot n^{(m-1)/2} / (mn-1)!\right] = \varphi(x)$

⇒ We notice that $\varphi(x)$ is a Constant independent of x.

2nd To find ① / ②, we notice that $\boxed{\varphi(1/m) = \varphi(x)}$ ⇒ to Prove this:

$\varphi(1/m) = m^{mx\cdot 1/m}\left[\prod_{r=0}^{m-1}\Gamma\left(\frac{1}{m}+\frac{r}{m}\right) / 1\right] = m\prod_{r=0}^{m-1}\Gamma\left(\frac{1+r}{m}\right) =$

$m\prod_{r=0}^{m-1}\left((n-1)!\cdot n^{(1+r)/m} / \prod_{k=0}^{n-1}\left(\frac{1+r}{m}+k\right)\right) =$

$\left[m\cdot m^{mn/2+mn/2}\cdot\left((n-1)!\right)^m\cdot n^{\sum_{r=0}^{m-1}(1+r)/m} / \prod_{r=0}^{m-1}\prod_{k=0}^{n-1}(1+r+mk)\right]$

but $\prod_{r=0}^{m-1}\prod_{k=0}^{n-1}(1+r+mk) = 1\cdot(m-1)\cdot(m)\cdots(1+mn-m+m-1)$

$= mn(mn-1)! = (mn-1)!\,mn$

also, $\sum_{r=0}^{m-1}(1+r) = 1+2+\cdots+m = \frac{m}{2}(1+m) = \frac{m}{2}(m+1) \Rightarrow$

Therefore, $\varphi(1/m) = n^{(m+1)/2}\cdot m^{mn}\cdot\left((n-1)!\right)^m / (mn-1)!\,mn$.

$n^{(m+1)/2} = n^{(m-1)/2}\cdot n$ (m term)

3rd ⑧ $\varphi(1/m) = m\,\Gamma(1/m)\cdot\Gamma(2/m)\cdot\cdots\cdot\Gamma\left(\frac{m-1}{m}\right)$

$\varphi(1/m) = m\,\Gamma(1-1/m)\cdot\Gamma(1-2/m)\cdot\cdots\cdot\Gamma(1/m)$

$\Rightarrow \left[\varphi(1/m)\right]^2 = m^2\prod_{r=1}^{m-1}\Gamma(r/m)\cdot\Gamma(1-r/m)$

but since $\Gamma(x)\cdot\Gamma(1-x) = \pi/\sin\pi x\ . \Rightarrow$

$\left[\varphi(1/m)\right]^2 = m^2\cdot\pi^{m-1}\cdot\left(1/\prod_{r=1}^{m-1}\sin r\pi/m\right)$. since $\left(\sin\frac{r\pi}{m}\right) \rightleftarrows$

since $\prod_{r=1}^{m-1}\sin(r\pi/m) = m/2^{m-1} \Rightarrow \varphi(1/m) = \sqrt{m}\cdot(2\pi)^{m-1}$.

① / ② $= \left[\left((n-1)!\right)^m\cdot m^{mn}\cdot n^{(m-1)/2} / (mn-1)!\right]$.

$\varphi(1/m) = n^{(m-1)/2}\cdot mn\cdot m^{mn}\cdot\left((n-1)!\right)^m / (mn-1)!\,mn \Rightarrow \varphi(x) = \varphi(1/m)$

$m+1/2 = (m-1+2)/2 =$

$(m-1)/2 + 1 \Rightarrow$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$\int f(x)\,dx$
$\sin x$	$\cos x$	$\sin^{-1}x$	$1/\sqrt{1-x^2}$	$\int \sin x\,dx = -\cos x$
$\cos x$	$-\sin x$	$\cos^{-1}x$	$-1/\sqrt{1-x^2}$	$\int \cos x\,dx = \sin x$
$\tan x$	$1/\cos^2 x = \sec^2 x$	$\tan^{-1}x$	$1/(1+x^2)$	$\int \tan x\,dx = -\ln\lvert\cos x\rvert = \ln\lvert\sec x\rvert$
$\cot x$	$-1/\sin^2 x = -\operatorname{cosec}^2 x$	$\cot^{-1}x$	$-1/(1+x^2)$	$\int \cot x\,dx = \ln\lvert\sin x\rvert$
$\sec x$	$\tan x/\cos x = \sec x\tan x$	$\sec^{-1}x$	$1/\lvert x\rvert\sqrt{x^2-1}$	$\int \sec x\,dx = \ln\lvert\sec x+\tan x\rvert$
$\operatorname{cosec} x$	$-\frac{\cot x}{\sin x} = -\operatorname{cosec} x\cot x$	$\operatorname{cosec}^{-1}x$	$-1/\lvert x\rvert\sqrt{x^2-1}$	$\int \operatorname{cosec} x\,dx = \ln\lvert\operatorname{cosec} x-\cot x\rvert$
$\sinh x$	$\cosh x$	$\sinh^{-1}x$	$1/\sqrt{x^2+1}$	$\int \sinh x\,dx = \cosh x$
$\cosh x$	$\sinh x$	$\cosh^{-1}x$	$1/\sqrt{x^2-1}$	$\int \cosh x\,dx = \sinh x$
$\tanh x$	$1/\cosh^2 x$	$\tanh^{-1}x$	$1/(1-x^2)$, $x^2<1$ (always)	$\int \tanh x\,dx = \ln(\cosh x)$
$\coth x$	$-1/\sinh^2 x$	$\coth^{-1}x$	$-1/(x^2-1)$, $x^2>1$ (always)	$\int \coth x\,dx = \ln\lvert\sinh x\rvert$
$\operatorname{sech} x$	$-\tanh x\operatorname{sech} x$	$\operatorname{sech}^{-1}x$	$-1/\lvert x\rvert\sqrt{1-x^2}$	$\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x)$
$\operatorname{cosech} x$	$-\coth x\operatorname{cosech} x$	$\operatorname{cosech}^{-1}x$	$-1/\lvert x\rvert\sqrt{1+x^2}$	$\int \operatorname{cosech} x\,dx = \ln\lvert\operatorname{cosech} x-\coth x\rvert$

c (constant) $\quad 0$	$\int dx/\cos^2 x = \tan x$	$\int c\,dx = cx$
$x^\alpha \quad \alpha\in\mathbb{R} \quad \alpha x^{\alpha-1}$	$\int dx/\sin^2 x = -\cot x$	$\int x^n dx = x^{n+1}/(n+1), \; n\neq -1$
$1/x \quad -1/x^2$	$\int dx/\cosh^2 x = \tanh x$	$\int dx/x = \ln\lvert x\rvert$
$e^x \quad e^x$	$\int dx/\sinh^2 x = -\coth x$	$\int e^x dx = e^x$
$a^x \quad a^x \ln a$	always add a constant to the integration.	$\int a^x dx = a^x/\ln a$

* $f(x)$, $g(x)$ two differentiable fns.

1. $\frac{d}{dx}(cf) = c\cdot f'$
2. $\frac{d}{dx}(f\pm g) = f'\pm g'$
3. $\frac{d}{dx}(f\cdot g) = f\cdot g' + f'\cdot g$
4. $\frac{d}{dx}\left(\frac{f}{g}\right) = (gf' - fg')/g^2, \; g\neq 0$

5. $g(y)$, $y=f(x)$ $\Rightarrow$ $\frac{d}{dx}g(y) = \frac{d}{dy}(g)\cdot\frac{dy}{dx}$
6. $f(x) = y \Rightarrow$ if $f^{-1}(y) = x$ i.e bijective mapping $\Rightarrow$ $\frac{d}{dy}[f^{-1}(y)] = 1/(df(x)/dx)$

7. **Leibniz's Rule for Differentiation**: $\frac{d^n}{dx^n}(f\cdot g) = \sum_{r=0}^{n} {}^nC_r\, f^{(n-r)}\cdot g^{(r)}$ where f and g are (n) times differentiable. ${}^nC_r = n!/r!(n-r)!$

8. $\sin x_1 - \sin x_2 = 2\sin\frac{x_1-x_2}{2}\cdot\cos\frac{x_1+x_2}{2}$; $\cos x_1 - \cos x_2 = -2\sin\frac{x_1-x_2}{2}\cdot\sin\frac{x_1+x_2}{2}$

$\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$. $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$

$\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$. $\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x|>1$

$f(x)$ is diff. at $[a,b]$, $x_0\in[a,b]$ $\Rightarrow$ Lagrange Mean Value theorem: $\frac{f(b)-f(a)}{b-a} = f'(x_0)$.

$g(x)$ is diff. at $[a,b]$ $\Rightarrow$ Cauchy M.V. theorem: $\frac{f(b)-f(a)}{g(b)-f(a)} = \frac{f'(x_0)}{g'(x_0)}$, $g'(x)\neq 0 \;\forall x\in(a,b)$

M.V. Theorem: $f(a+h) = f(a) = h\cdot f'(a+\theta h)$, $a+h=b$, $0<\theta<1$.

$\sin x$ = opposite/hypotenuse ; $\cos x$ = adjacent/hypotenuse ; $\cot x = \frac{1}{\tan x}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$

(4) B

* L'hôpital's Rule: If $\lim_{x\to c} g(x) = 0$, $\lim_{x\to c} f(x) = 0 \Rightarrow$ $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$. Also used for the form $\frac{\infty}{\infty}$.

* Taylor's Theorem: $f(x) = \sum_{r=0}^{n} f^{(r)}(x_0) \frac{(x-x_0)^r}{r!} + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$

f is differentiable (n+1) times in the interval (a,b) ; $x, x_0 \in (a,b)$, $x_0 < c < x$

* Rules for carring out integrations.

1. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ notice, the integrand possesses a good form that can be related to direct derivation of the function $\ln|f(x)|$ which is the value of the integral in addition to a constant c.
2. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$
3. Integration By Parts: $\int u\, dv = uv - \int v\, du$.
4. " " Substitution: If the integrand has a form that is difficult to integrate directly $\Rightarrow$ we can transform $f(x)$ into $f(\varphi(t))$ s.t $\int f(x)\, dx = \int f(\varphi(t))\, dt$ s.t $f(\varphi(t))$ has a more easier form that can be related to standard functions in tables of differentiation.
5. Integration By Partial Fractions:
6. Integratinon By trigonometric Substitutions:

If the integrand $f(x)$ is a function of $\sin x$, $\cos x \Rightarrow$ using the substitution $\tan\frac{x}{2} = t \Rightarrow \int f(x)\, dx = \int f(\varphi(t))\, dt$

and $\sin x = 2\tan(\frac{x}{2}) / 1 + \tan^2(x/2) = 2t / 1+t^2$ (1)

$\cos x = 1 - \tan^2(\frac{x}{2}) / 1 + \tan^2(x/2) = 1 - t^2 / (1+t^2)$ (2)

$\tan x = 2\tan(x/2) / 1 - \tan^2(x/2) = 2t / 1-t^2$ (3)

also $x = 2\tan^{-1} t \Rightarrow dx = \frac{2\,dt}{1+t^2}$.

7. Recurrence relations to carry out some integrations:

$I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots \frac{1}{2} I_{0 \text{ or } 1}$

i.e $I_n = \frac{n-1}{n} I_{n-2}$ $n \in Z^+$

$S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$; $S_n = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

In the 1st table of integrations one must notice some symmetries especially with regart to 1) sign 2) alternation and arraigement of sign. 3) the f^n used to carry out the integration. 4) regularity in the arrangement of $f^{n's}$ representing the derivatives.

+ Find y' for :- (1) $y = (1+2x)^{\sin x}$ (2) $y = (1-x^2/1+x^2)^{5x}$

Integration of Products and Quotients of two fns.

* If the integrand is a quotient which is difficult to differentiate, we turn the quotient into a product and get y' by differentiating the product.

1. If we put $(x+b)$ instead of x, where b = Const. =>

$$\int (x+8)^{1/4}\, dx = \frac{(x+8)^{1+1/4}}{1+1/4} = (x+8)^{5/4} / 5/4 .$$

2. If we put ax instead of x ; a = Const., remember to divide by a i.e. the Coefficient of x. =>

$$\int \cos 2x\, dx = \sin 2x/2 . \quad \int \operatorname{sech}^2(x/4)\, dx = 4 \tanh(x/4) .$$

3. If we put $ax+b$ instead of x, remember to divide by a i.e. the Coefficient of x. =>

$$\int dx/(2-5x) = (-1/5) \ln(2-5x)$$

$$\int dx/\sqrt{a^2-x^2} = \int d(x/a)/\sqrt{1-(x/a)^2} = \sin^{-1}(x/a) .$$

$$\int dx/\sqrt{a^2+x^2} = \sinh^{-1}(x/a) .$$

Important Identities :

* $\cos^2 x + \sin^2 x = 1$ but $\cosh^2 x - \sinh^2 x = 1$
* $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$; $\sinh 2x = 2 \sinh x . \cosh x$
  $2\cosh^2 x - 1 = \cosh 2x = 2\sinh^2 x + 1$
* $\operatorname{Cosec}^{-1}(1/x) = \sin^{-1} x$ or $\operatorname{Cosec}^{-1} x = \sin^{-1}(1/x)$
* $\operatorname{Cosech}^{-1}(1/x) = \sinh^{-1} x$ or $\operatorname{Cosech}^{-1} x = \sinh^{-1}(1/x)$
* $\operatorname{Coth}^{-1}(1/x) = \tanh^{-1} x$ or $\operatorname{Coth}^{-1} x = \tanh^{-1}(1/x)$

Solve: $\tan^{-1} a + \tan^{-1} b = \tan^{-1}[(a+b)/(1-ab)]$.

Let $\tan^{-1} a = \alpha$, $\tan^{-1} b = \beta$ => $\tan\alpha = a$, $\tan\beta = b$ =>

$\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha . \tan\beta)$ ■

By common sense

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1}[(a-b)/(1+ab)] .$$

1. $\int [f(x)]^n f'(x)\, dx = [f(x)]^{n+1}/(n+1)$. 2. $\int f'(x)\, dx / f(x) = \ln[f(x)]$

3. $2\int \frac{1}{2} \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}$. $n = -1/2$

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Solve: $\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}\,dx = \ln(\sec x+\tan x)$

$\int \sec x\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{1}{1-\sin^2 x}\cdot\cos x\,dx = \int \frac{1}{1-\sin^2 x}(\sin x)'\,dx = \tanh^{-1}(\sin x)$

Solve: $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{(\operatorname{cosec} x-\cot x)}\,dx = \ln(\operatorname{cosec} x-\cot x)$.

$\int \operatorname{cosec} x\,dx = \int \frac{\sin x}{\sin^2 x}\,dx = -\int \frac{1}{1-\cos^2 x}\cdot(-\sin x)\,dx = -\tanh^{-1}(\cos x)$

Solve: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int -\frac{\sin x}{\cos x}\,dx = -\ln\cos x = \ln(\sec x)$.

Solve: $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln(\sin x)$. «Notice the symmetry»

Solve: $\int \tanh x\,dx$ and $\int \coth x\,dx$ «are solved by the same method»

Solve: $\int \operatorname{sech} x\,dx = \int \frac{1}{\cosh x}\,dx = \int \frac{\cosh x}{\cosh^2 x}\,dx = \int \frac{\cosh x}{1+\sinh^2 x}\,dx =$

From here we notice that $[\tan^{-1} y]' = \frac{1}{1+y^2}\cdot y'$, taking $y = \sinh x \Rightarrow$

$[\tan^{-1} y]' = \tan^{-1}(\sinh x) = \frac{y'}{1+y^2} = \frac{1}{1+\sinh^2 x}\cdot(\sinh x)' = \frac{\cosh x}{1+\sinh^2 x}$

Therefore, $\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x)$.

Solve: $\int \operatorname{cosech} x\,dx = \int \frac{1}{\sinh x}\,dx = \int \frac{\sinh x}{\sinh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x-1}\,dx =$

$-\int -\frac{1}{\cosh^2 x-1}\cdot\sinh x\,dx \Rightarrow$ It is noticed that if we take

$\cosh x = y \Rightarrow = \int \left(-\frac{1}{y^2-1}\right)\cdot y'\,dx = -\int (d\coth^{-1} y/dx)\,dx$

since $[\coth^{-1} y]' = -\frac{1}{y^2-1}\cdot y'$. Therefore,

$\int \operatorname{cosech} x\,dx = -\coth^{-1}(\cosh x)$.

Another solution is, $\int \frac{\operatorname{cosech} x-\coth x}{\operatorname{cosech} x-\coth x}\cdot\operatorname{cosech} x\,dx = \ln|\operatorname{cosech} x-\coth x|$.

Another Solution for $\int \operatorname{sech} x\,dx$:

Let $f(x) = \ln(\operatorname{sech} x+i\tanh x) \Rightarrow \frac{df}{dx} = \frac{1}{\operatorname{sech} x+i\tanh x}\cdot(-\operatorname{sech} x\tanh x+i\operatorname{sech}^2 x)$

$\int df = i\int -i\,\frac{-\operatorname{sech} x\tanh x+i\operatorname{sech}^2 x}{\operatorname{sech} x+i\tanh x}\,dx = i\int \frac{\operatorname{sech} x(i\tanh x+\operatorname{sech} x)}{(\operatorname{sech} x+i\tanh x)}\,dx =$

$i\int \operatorname{sech} x\,dx = \int df = f(x) = \ln(\operatorname{sech} x+i\tanh x) \Rightarrow$

$\int \operatorname{sech} x\,dx = -i\ln(\operatorname{sech} x+i\tanh x)$.

notice $d(ix)/dx = i$, $d(ix)/d(ix) = 1$

The integral's solution was found easily once we have prepared the appropriate identity, and the iterated (−) sign is used with the part of the integrand that balances it s.t it takes the form of $\int \frac{dF(x)}{dx}\,dx = F(x)$ where $F(x)$ is one of the known standard forms of differentiable functions.

Tables of the integrals of Square trigonometric and hyperbolic functions:

$\int \sin^2 x\,dx = \frac{1}{2}\int(1-\cos 2x)\,dx = \frac{1}{2}(x - \frac{\sin 2x}{2})$	$\int \sinh^2 x\,dx = \frac{1}{2}\int(\cosh 2x - 1)\,dx = \frac{1}{2}(\frac{\sinh 2x}{2} - x)$
$\int \cos^2 x\,dx = \frac{1}{2}\int(1+\cos 2x)\,dx = \frac{1}{2}(x + \frac{\sin 2x}{2})$	$\int \cosh^2 x\,dx = \frac{1}{2}\int(\cosh 2x + 1)\,dx = \frac{1}{2}(\frac{\sinh 2x}{2} + x)$
$\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x$	$\int \tanh^2 x\,dx = \int(1 - \operatorname{sech}^2 x)\,dx = x - \tanh x$
$\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 x - 1)\,dx = -\cot x - x$	$\int \coth^2 x\,dx = \int(1 + \operatorname{cosech}^2 x)\,dx = x - \coth x$
$\int \sec^2 x\,dx = \tan x$	$\int \operatorname{sech}^2 x\,dx = \tanh x$
$\int \operatorname{cosec} x\,dx = -\cot x$	$\int \operatorname{cosech}^2 x\,dx = -\operatorname{cotanh} x$

Integration Rules:

1. $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}[x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}]$

Let $x = a\sin u$ also try $x = a\tanh\theta$

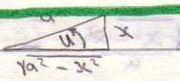
2. $\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}[x\sqrt{a^2+x^2} + a^2 \sinh^{-1}\frac{x}{a}]$

Let $x = a\sinh x$ also try $x = a\tan\theta$ or $a\cot\theta$

3. $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}[x\sqrt{x^2-a^2} - a^2 \cosh^{-1}\frac{x}{a}]$

Let $x = a\cosh u$ also try $x = a\sec\theta$ or $a\operatorname{cosec}\theta$

1. Let $x = a\sin u$ i.e $x/a = \sin u$

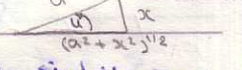


$dx = a\cos u\,du \Rightarrow \int \sqrt{a^2 - a^2\sin^2 u} = \int a\cos u \cdot a\cos u\,du =$

$a^2\int \cos^2 u\,du = a^2 \cdot \frac{1}{2}(u + \frac{\sin 2u}{2}) = \frac{a^2}{2}[u + \sin u\cos u] =$

$\frac{a^2}{2}[\sin u\sqrt{1-\sin^2 u} + u] = \frac{a^2}{2}[\frac{x}{a}\sqrt{1-(\frac{x}{a})^2} + \sin^{-1}\frac{x}{a}]$.

2. Let $x = a\sinh u \Rightarrow dx = a\cosh u\,du$



$\int \sqrt{a^2 + a^2\sinh^2 u} \cdot a\cosh u\,du = \int a^2\cosh^2 u\,du = a^2 \cdot \frac{1}{2}(\frac{\sinh 2u}{2} + u)$

$= \frac{a^2}{2}[\sinh u \cdot \cosh u + u] = \frac{a^2}{2}[\frac{x}{a}\sqrt{1+(\frac{x}{a})^2} + \sinh^{-1}\frac{x}{a}]$

3. Let $x = a\cosh u \Rightarrow dx = a\sinh u\,du \Rightarrow$

$\int \sqrt{a^2\cosh^2 u - a^2} \cdot a\sinh u\,du = a^2\int \sinh^2 u\,du =$

$\frac{a^2}{2}(\frac{\sinh 2u}{2} - u) = \frac{a^2}{2}(\cosh u \cdot \sinh u - u) =$

$\frac{a^2}{2}[\frac{x}{a}\cdot\sqrt{(\frac{x}{a})^2 - 1} - \cosh^{-1}\frac{x}{a}]$

* Both trigonometric and hyperbolic substitutions are equally correct to use to solve such a kind of integrals, only the length of the solution that suggests one substitution to be prefered to the other one.

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Integration by Parts : $\int u\,dv = uv - \int v\,du$

Solve : $I = \int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$, but $d(\tan x) = \sec^2 x\,dx \Rightarrow$

$= \int \sec x \cdot d(\tan x) = \tan x \cdot \sec x - \int \tan^2 x \cdot \sec x\,dx =$

$\tan x \cdot \sec x - \int (\sec^2 x - 1)\sec x\,dx = \sec x \cdot \tan x - I + \int \sec x\,dx \Rightarrow$

$\frac{1}{2} \times 2I = [\sec x \cdot \tan x + \ln|\sec x + \tan x|] \times \frac{1}{2}$.

Solve : $\int \sqrt{a^2 - x^2}\,dx$ let $x = a\tanh\theta \Rightarrow a^2 \int \operatorname{sech}\theta \cdot \operatorname{sech}^2\theta\,d\theta = I$

$= \tanh\theta \cdot \operatorname{sech}\theta + \int \tanh^2\theta \cdot \operatorname{sech}\theta\,d\theta = \int (1 - \operatorname{sech}^2\theta)\operatorname{sech}\theta\,d\theta) \Rightarrow$

$2I = a^2[\tanh\theta \cdot \operatorname{sech}\theta + \int \operatorname{sech}\theta\,d\theta] = a^2[\tanh\theta \cdot \operatorname{sech}\theta + \tan^{-1}(\sinh\theta)] \Rightarrow$

$2I = a^2\left(\frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2} + \tan^{-1}\left[(\cosh\theta) \cdot \frac{x}{a}\right]\right) = \frac{x}{a}\sqrt{a^2 - x^2} + a^2 \tan^{-1}\left[\frac{x}{a} \cdot \frac{a}{(a^2 - x^2)^{1/2}}\right]$

Solve : $\int \sqrt{a^2 + x^2}\,dx$ let $x = a\tan\theta \Rightarrow a^2 \int \sec^2\theta \cdot d(\tan\theta) =$

$\frac{a^2}{2}(\sec\theta \cdot \tan\theta + \ln|\sec\theta + \tan\theta|) = \frac{a^2}{2}\left[\frac{x}{a^2}\sqrt{a^2 + x^2} + \ln\left(\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right)\right]$

Solve : $\int \sqrt{x^2 - a^2}\,dx$ let $x = a\sec\theta \Rightarrow a^2 \int \sqrt{\sec^2\theta - 1} \cdot d(\sec\theta) =$

$a^2 \int \tan^2\theta \cdot \sec\theta\,d\theta = \frac{1}{2}a^2 \tan\theta \cdot \sec\theta - \frac{1}{2}a^2 \ln|\sec\theta + \tan\theta| =$

$\frac{1}{2}a^2\left(\frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \ln\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right)$.

Differential Equations of the 1st order and the 1st degree :

I. Separation of Variables :

Which is used when we can write the diff. Eqtn as

$f(x)\,dx = g(y)\,dy$

Ex. Solve $2(x+1)\frac{dy}{dx} - 3(y+2) = 0$

$\Rightarrow \frac{1}{2}\int \frac{dx}{x+1} = \frac{1}{3}\int \frac{dy}{y+2} \Rightarrow \frac{1}{2}\ln(x+1) = \frac{1}{3}\ln(y+2) + \ln c$

$\ln(y+2)^2 = \ln[(x+1)^3 c] \Rightarrow (y+2)^2 = c(x+1)^3$.

II. Homogeneous Equations :

Which can be written as $\frac{dy}{dx} = f(y/x)$ ①

Using the substitution $v = y/x \Rightarrow y = xv$ ②

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ ③ , from ②, ③ substituting in ① $\Rightarrow$

$x\frac{dv}{dx} + v = f(v)$ which can be solved by separating the variables $v, x \Rightarrow x\frac{dv}{dx} = f(v) - v \Rightarrow$

$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} = \ln x$ ...

After the integral is carried out $\Rightarrow$ we substitute v by its value (y/x).

Solve: $(xy + x^2)\frac{dy}{dx} + y^2 = 0$ let $(y/x = v)$

$$\frac{dy}{dx} = -\frac{y^2}{x(x+y)} = -\frac{(y/x)^2}{1+(y/x)} = \frac{-v^2}{1+v} = x\frac{dv}{dx} + v \Rightarrow$$

$$\left(\frac{-v^2}{1+v} - v\right) = x(dv/dx) \Rightarrow \frac{v^2 - v - v^2}{1+v} = x\frac{dv}{dx} \Rightarrow$$

$$\int \frac{1+v}{v(2v+1)}\,dv = \int \frac{dx}{x} \Rightarrow$$

$$-\frac{v(2v+1)}{1+v} = x\frac{dv}{dx} \Rightarrow -\int \frac{dx}{x} = \int \frac{1+v}{v(2v+1)}\,dv$$

$$-\ln x + \ln c = \int \left(\frac{a}{v} + \frac{b}{2v+1}\right) dv \Rightarrow$$

$$\frac{2av + a + bv}{(2v+1)v} = \frac{v+1}{v(2v+1)} \Rightarrow (2a+b)v + a = v + 1 \Rightarrow$$

$$a = 1,\ b + 2a = 1 \Rightarrow b = -1 \Rightarrow$$

$$\ln\frac{c}{x} = \int \frac{dv}{v} - \frac{1}{2}\int \frac{2\,dv}{2v+1} = \ln v - \frac{1}{2}\ln(2v+1)$$

$$= \ln\left(\frac{v}{(2v+1)^{1/2}}\right) \Rightarrow \frac{c}{x} = \frac{v}{\sqrt{2v+1}} = \frac{y/x}{\sqrt{2(y/x)^2+1}} \Rightarrow$$

$$c = y/\sqrt{2(y/x)^2+1}.$$

III. Exact Equations:

We say that, $P\,dx + Q\,dy$ is a total differential if we find a function $u = u(x,y)$ s.t

$$P\,dx + Q\,dy = du, \quad P = P(x,y),\ Q = Q(x,y).$$

We notice

+ $d(yx) = y\,dx + x\,dy$ "Total Diff." (1)
+ $y\,dx - x\,dy$ isn't a total Diff. (2)
+ $\frac{y\,dx - x\,dy}{y^2} = \frac{dx}{y} - \frac{x}{y^2}dy = d\frac{x}{y}$ (3) "Total Diff."

notice $(1/y^2)$ in (3) is for (2) an "integrating factor".

When does $P\,dx + Q\,dy$ becomes a total Differential?

A necessary and sufficient Condition for $P\,dx + Q\,dy$ to become a total differential

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$, and Coefficients are inseparable

"i.e we can not separate the variables."

Necessary Condition:

Let $P\,dx + Q\,dy = du$ = total diff. (a)

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du \quad (2)$$

Since x, y are independent $\Rightarrow$ y may chosen Const. $\Rightarrow$

$dy = 0 \Rightarrow \frac{\partial u}{\partial x} = P$. $\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (I) also, $\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ (II)

Since (I) = (II).

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$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is a necessary condition "

$\Rightarrow$ Inversely, Prove $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\exists$ a function s.t $P dx + Q dy = du$; where in this case, this is called an ~~Exact~~ " Exact Equation "

$P dx + Q dy = du = 0$ i.e $u = $ Const.

+ How to find u?

$$u = \int_{x_0}^{x} P(x, y) dx + \int_{y_0}^{y} Q(x_0, y) dy$$

x_0 is an arbitrary Constant which is chosen s.t it simplifies the function and therefore simplifies the integration.

Solve: $(12x + 5y - 9) dx + (5x + 2y - 4) dy = 0$

$\partial P/\partial y = 5$, $dQ/dx = 5$ = exact diff.

1. Condition of exact differential.
2. Find function (s) which yields an exact diff.
3. $u = $ Const.

$u = \int_0^x (12x + 5y - 9) dx + \int_0^y (5x_0 + 2y - 4) dy \Rightarrow$

$u = 6x^2 + 5xy - 9x \Big|_0^x + y^2 - 4y \Big|_0^y = 6x^2 + 5xy - 9x + y^2 - 4y = C$

IV Linear Equations (in y or in x)

1st degree, 1st order $(D + P(x))y = Q(x)$

$\frac{dy}{dx} + P(x) y = Q(x)$ " Linear eqt^n of y "

Two Conditions are imposed:

(1) Find $\int P dx$ where P is Coefficient of y

(2) Find $e^{\int P dx}$. In (1), the integral is usually found in terms of $\ln$ a function.

Solution: $y\, e^{\int P dx}$

$e^{\int P dx}$ which is called the integrating factor

$y \underbrace{e^{\int P dx}}_{\text{(L.H.S)}} = \int \underbrace{Q e^{\int P dx}}_{\text{(R.H.S)}} dx$. Notice the Coefficient of $\frac{dy}{dx}$ is unity, if not, always Convert it into the standard form above.

Solve: $\frac{dy}{dx} + \left(\frac{x}{1+x^2}\right) y = x$

$\int P dx = \int \frac{x}{1+x^2} dx = \ln (1+x^2)^{1/2}$.

$y \exp \ln \sqrt{1+x^2} = \int x \sqrt{1+x^2} dx$

$y \sqrt{1+x^2} = \frac{1}{2} \int (1+x^2)^{1/2} \cdot 2x\, dx = \frac{1}{2} \frac{(1+x^2)^{3/2}}{3/2} + C$

Notice, the same for " $\frac{dx}{dy} + R(y) x = S(y)$ "

V Equations that may reduce to homogeneous equations:

"the two lines dose not pass through the origin."

* $\frac{dy}{dx} = f\left(\frac{ax+by}{a'x+b'y}\right)$

① (i) $a/a' \neq b/b' \Rightarrow$

$\frac{dy}{dx} = f\left(\frac{a+b(\frac{y}{x})}{a'+b'(\frac{y}{x})}\right) = F\left(\frac{y}{x}\right)$ which is homogeneous

$\Rightarrow$ put $y/x = v$

(ii) $a/a' = b/b'$, $a'/a = b'/b = \lambda$

$a' = \lambda a$, $b' = \lambda b$

(i) implies $\frac{dy}{dx} = f\left(\frac{ax+by}{\lambda(ax+by)}\right) = f\left(\frac{1}{\lambda}\right) =$ const.

② $ax + by = 0$

$a'x + b'y = 0$

[in the Case of the two lines are parallel.]

* Another Case: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{a'x+b'y+c'}\right)$

when the two lines are not passing the origin.

Solve: $(12x+5y-4)\,dy + (5x+2y-4)\,dx = 0$

we can follow the method of solving this problem and take it as a standard method.

$\frac{dy}{dx} = \frac{-(12x+5y-9)}{(5x+2y-4)}$

① find the points of intersection $\Rightarrow (x,y)_{\text{intersection}} = (2,-3)$

② put $x = x+2$, $y = y-3 \Rightarrow$

$\frac{dy}{dx} = -\frac{12x+5y}{5x+2y} = -\frac{12+5(y/x)}{5+2(y/x)}$ which is homogeneous

let $y/x = v \Rightarrow dy/dx = -\frac{12+5v}{5+2v} = v + dv/dx$

$\frac{dv}{dx} = \frac{12+5v}{5+2v} - v = \frac{12-2v^2}{5+2v} \Rightarrow \frac{1}{2}\int \frac{5+2v}{6-v^2}\,dv = \int dx = x + c$

$\frac{1}{2}\left[\frac{5}{6\sqrt{6}}\int_0^{\sqrt{6}} \frac{dv}{1-(v/\sqrt{6})^2} + 2\int \frac{-2v\,dv}{6-v^2}\right] = \frac{1}{2}\left[\frac{5}{6\sqrt{6}}\tanh^{-1}(v/\sqrt{6}) - \ln(6-v^2)\right]$

$\frac{1}{2}\left[\frac{5}{6^{3/2}}\ln\left(\frac{(1+v/\sqrt{6})}{1-v/\sqrt{6}}\right) - \ln(6-v^2)\right]$.

* $z = ax + by \Rightarrow \frac{dz}{dx} = a + b\frac{dy}{dx}$

and $\frac{dy}{dx} = f\left(\frac{ax+by+c}{\lambda(ax+by)+c'}\right) \Rightarrow \frac{dz}{dx} = a + bf\left(\frac{z+c}{\lambda z+c'}\right) = F(z)$

VI Equations that may be reduced to linear eqtns (Bernoulli's eqtns)

$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ ① $\Rightarrow y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$ ②

let $Y = y^{1-n} \Rightarrow dY/dx = (1-n)\,y^{(1-n-1)}\,dy/dx \Rightarrow$ $(X = x)$

$\frac{dY}{dx} = (1-n)\left(y^{-n}\frac{dy}{dx}\right)$ ③ $\cdot \frac{dY}{dx} + (1-n)\,P(x)\,Y = (1-n)\,Q(x)$,

$\left[dY/dX + P^*(x)\,Y = Q^*(x)\right]$ which is a linear eqtn in Y; put $Y = f(y) \Rightarrow$

$dY/dx = P(x)\,Y = Q(x)$.

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+ $\int_a^b f(x)\,dx$ or $a\int_a^b f(x)\,dx$; $-\int -f(x)\,dx$; $\int f(x)\cdot 1\,dx = \int f(x)\cdot\frac{g'(x)}{g'(x)}\,dx$ or

Observation: $\int f(x)\cdot 1\,dx = \int f(x)\cdot g'(x)\,dx$ [$g'(x) = 1$ identity]

Good substitution space lead to standard form of the integrand space s.t the integral is found easily.

+ What made it lead to a good form?

[Substitution space (operates on) Integrand space] $\overset{\text{gives}}{\Rightarrow}$ Good transformed space.

+ How, integrand space properties are looked for s.t good substitution space is used??

Of Course we have to know what properties the integrand possesses, or otherwise we have to establish the "general rules" by which we can determine the general properties any function space can possess s.t we can compare the integrand space to these general properties to decide what substitution space we have to use s.t we obtain the new good integrand, and explains Why we have chosen the particular substitution space, of Course, in order to reappear or transform the integrand space into the new good space.

i.e We have eliminated the confusing part of the integrand that prevents from carring out the integration operation directly on it.

Of Course one means good w.r.t to the overall capabilities of the class of function spaces, we deal with, to carry out the integration operation.

Substitution is: $x = s(u)$ or $u = \varphi(x)$

Hence, given the integrand space $f(x)$ we can relate it to some class of function spaces, which in turn, i.e the class, possesses some properties, therefore we can transform them, using the proper substitution space, into the good integrand space or into the standard form of their class of function space.

Lastly, the integration operation can be carried out easily on the acceptible integrand space.

+ $\int f(x)\,dx = x$; $\int \frac{f(x)}{g(x)}\,dx$; $\int dx / \sqrt{g(x)}$; $\int \sqrt{f}\,dx$; $\int \sqrt{f(x)/g(x)}\,dx$;

$\int (\sqrt{g(x)})^{-1/2}\,dx$, $\int [h(x)]^{l}\cdot[f(x)]^{m}\cdot[g(x)]^{n}\,dx$ $m,n \in \mathbb{Q}$ i.e fract. or int.

+ $\int f(x)\cdot g(x) = \int f(x)\,d(f(x))$. Integration by parts $\equiv$ Choose of the good parts.

Solve: If $y = (1+2x)^{\sin x}$ ⇒ get y'.

$\ln y = \sin x \ln(1+2x)$ ⇒ $\frac{y'}{y} = \sin x \cdot \frac{2}{1+2x} + \ln(1+2x)\cos x$

⇒ $y' = (1+2x)^{\sin x}\left[\frac{2\sin x}{1+2x} + \cos x \cdot \ln(1+2x)\right]$

Solve: If $y = \left(\frac{1-x^2}{1+x^2}\right)^{5x}$ ⇒ get y'

$\ln y = 5x \ln\left[\frac{(1-x^2)}{(1+x^2)}\right] = 5x[\ln(1-x^2) - \ln(1+x^2)]$

$\frac{y'}{y} = 5x\left[\frac{-2x}{1-x^2} - \frac{2x}{1+x^2}\right] + 5\ln\left(\frac{1-x^2}{1+x^2}\right)$ ⇒

$y' = \left[\frac{(1-x^2)}{(1+x^2)}\right]^{5x} \times 5x\left[\frac{-2x}{(1-x^2)} - \frac{2x}{(1+x^2)}\right] + 5\ln\left(\frac{1-x^2}{1+x^2}\right)$.

Get y' if $y = \sinh^2 mx \cdot \cosh mx$.

$y' = \sinh^2 mx \times m\sinh mx + \cosh mx \times 2\sinh mx \cdot \cosh mx \times m$ ⇒

$y' = m\sinh^3 mx + 2m\sinh mx \cdot \cosh^2 mx$.

Get y' if $y = \tanh^{-1}(\tan x/2)$ ⇒

$\tanh y = \tan(x/2)$ ⇒ $\frac{y'}{1-y^2} = \frac{1/2}{1+(x/2)^2}$

⇒ $y' = \frac{1}{2}(1-y^2)/[1+(x/2)^2]$.

If $y = (\sinh^{-1} x)^2$ show that $(1-x^2)y'' + xy' = 2$

$\sinh^{-1} x = \sqrt{y}$ ⇒ $\sinh y^{1/2} = x$ ⇒

$\cosh y^{1/2} \cdot \frac{1}{2} y^{-1/2} \cdot y' = 1$ ⇒ $\frac{1}{2}\cosh y^{1/2} \cdot y' \cdot y^{-1/2} = 1$

(or $2y^{1/2} = y' \cosh y^{1/2}$ ⇒ $[y^{-1/2} y' = y' \cdot \sinh y^{1/2} \cdot \frac{1}{2} y^{-1/2} \cdot y' +$

$\cosh y^{1/2} \cdot y'']$ ⇒ $y^{-1/2} y' - xy'^2/2 = y'' \sqrt{1+x^2}$

$y' = 2y^{1/2}/\sqrt{1+x^2}$, $y^{1/2} y'' \sqrt{1+x^2} = y' - (x y^{1/2} y'^2)/2$

$2y' - x y^{1/2} y'^2 = 2y^{1/2} \cdot y'' \sqrt{1+x^2}$

Evaluate: $\int_0^{\pi/2} \sin x \cdot \cos^4 x \, dx = -4\cos^4 x \Big|_0^{\pi/2} - \int_0^{\pi/2} \cos^5 x \, dx$

$I = \int_0^{\pi/2} \cos^5 x \, dx = \frac{4}{5} I_3 = \frac{4}{5}\cdot\frac{2}{3} I_1 = \frac{8}{15}\int \cos x \, dx = \frac{8}{15}\sin x \Big|_0^{\pi/2} = \frac{8}{15}$

Evaluate: $\int x(1+x^2)\,dx / \sqrt{1+x^2}$ ⇒ let $x = \tan u$ ⇒

$dx = \sec^2 u \, du$ ⇒ $(1+\tan^2 u)^{1/2} = \sec u$ ⇒ $\int \tan u \cdot \sec^2 u \cdot du =$

$\int \sec^2 u \cdot \tan u \cdot \sec u \, du = \int \sec^2 u \cdot d(\sec u) = \int y^2 \, dy =$

$\frac{1}{2} y^2 = \frac{1}{3}\sec^2 u = \frac{1}{3}(1+\tan^2 u)\sec u = \frac{1}{3}(1+x^2)\sqrt{1+x^2}$.

(14) B

Evaluate: $\frac{1}{\sqrt{3}}\int \sqrt{3}\cosh x\, dx / 1+(\sqrt{3})^2\sinh^2 x = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\sinh x)$.

Evaluate: $(1+x^2)\frac{dy}{dx} + xy = x(1+x^2)$

$dy + \frac{x}{1+x^2} y\, dx = x\, dx \Rightarrow$

$\frac{1}{2}\int \frac{2x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2) \Rightarrow$

$y(1+x^2)^{1/2} = \frac{1}{2}\int 2x(1+x^2)^{1/2} dx = \frac{1}{2}\int (1+x^2)^{1/2} d(x^2)$

$= \frac{1}{2}\int (1+u)^{1/2} du = \frac{1}{2}\int t^{1/2} dt = \frac{1}{2}\cdot\frac{1}{3/2}\cdot t^{3/2} = \frac{2}{3}t^{3/2}$

$\Rightarrow y\sqrt{1+x^2} = \frac{1}{3}(1+x^2)^{3/2}$.

Evaluate: $\frac{dy}{dx} + \tan x\, y = \sec x \Rightarrow$

$\int \tan x\, dx = -\ln|\cos x| \Rightarrow$

$y\sec x = \int \sec^2 x\, dx = \tan x \Rightarrow y = \cos x \tan x = \sin x$.

1 (a) Find y' given $y = (1-x)^{2x}$

b. If $y = (\sin^{-1}x)^2$, show that $(1-x^2)y'' - xy' = 2$

c. Use two different methods to evaluate $\int_0^{\pi/2} (2\sin 2\theta)^4 \cos^2\theta\, d\theta$.

d. Solve $y' + y\cot x = \operatorname{cosec}^3 x$.

2 Solve (i) $(D^4 + 5D^2 + 4)y = e^x + e^{-x}$

(ii) $(D^4 + 18D^2 + 81)y = 0$.

(iii) $(D^2 + 9)y = \sec 3x$

(iv) $(D+1)^2 y = xe^{2x} + x^2$

3 (i) Use Euler's definition to show that $\Gamma(x+1) = x\Gamma(x)$.

Can you obtain the same result by using the relation

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

(ii) State and prove Legendre's Duplication formula for $\Gamma(2x)$, and show that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

(iii) By using the Beta function, show that

$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \pi/\sqrt{2}$.

(iv) Define Bessel function and show that

$\frac{d}{dx}(x^n J_n(x)) = -x^n J_{n+1}(x)$.

(15) B

Solve: $I=\int \frac{dx}{3-2\cos x}=\int \frac{\frac{1}{\cos x}\,dx}{\frac{3}{\cos x}-2}=\int \frac{\sec x\,dx}{3\sec x-2}=\int \frac{\sec x\tan x\,dx}{(3\sec x-2)\tan x}=$

$\int \frac{d\sec x}{(3\sec x-2)(\sqrt{\sec^2 x-1})} \Rightarrow$ let $\sec x=t \Rightarrow I=\int \frac{dt}{(3t-2)\sqrt{t^2-1}}$

let $t=au+b$ s.t $3au+3b-2=3au$ i.e $b=2/3$ and

$\frac{t^2-1}{(cu)^2}=1-\left(\frac{d-eu}{cu}\right)^2=\frac{a^2u^2+2abu+b^2-1}{(cu)^2}=\frac{(c^2-e^2)u^2+2edu-d^2}{c^2u^2}$

$a^2=c^2-e^2$; $\frac{4}{3}a=+2ed$; $b^2-1=-d^2$ $\Rightarrow \frac{4}{5}-\frac{9}{9}=+\frac{5}{9}=+d^2$

$\frac{16}{9}a^2=4c^2d^2=+\frac{20}{9}e^2 \Rightarrow 20a^2=20c^2-20e^2 \Rightarrow 20a^2+16a^2=36a^2=20c^2$

$4\times 9\,b^2=4\times 5c^2 \Rightarrow c=(3/\sqrt{5})a$; $16a^2=20e^2 \Rightarrow e=(2/\sqrt{5})a$; $d=\sqrt{5}/3$

$c=(3/\sqrt{5})a$; $b=\frac{2}{5}$; $d=\sqrt{5}/3$; $e=(2/\sqrt{5})a$;

given (a) we can determine c and e. $\Rightarrow dt=a\,du$

$\Rightarrow \int \frac{dt}{(3t-2)\sqrt{t^2-1}}=\int \frac{a\,du}{3au\sqrt{(au+\frac{2}{3})^2-1}}=\int \frac{a\times 1}{\sqrt{\frac{(3au+2)^2-9}{9}}}\cdot\left(\frac{1}{3au}\right)du$

$=-\int \frac{3au}{\sqrt{9a^2u^2+12au+4-9}}\left(-\frac{1}{3au^2}\right)du=-\int \frac{\sqrt{5}x\,1/3\times 1}{\sqrt{5}\sqrt{\frac{9a^2u^2+12au-5}{9\times 9a^2u^2}}}\cdot\left(-\frac{3a}{9a^2u^2}\right)du$

$1-\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)^2=\left(\frac{9}{5}a^2-\frac{4}{5}a^2\right)u^2+2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{\sqrt{5}}\right)au-\frac{5}{9}=$

$\frac{1}{(3/5)(au)^2}\left(a^2u^2+\frac{4}{3}au-\frac{5}{9}\right)=\frac{(9a^2u^2+12au-5)\times 5}{(9au)^2}$

$\Rightarrow I=-\frac{\sqrt{5}}{3}\int \frac{1}{\sqrt{1-\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)^2}}\cdot\left(-\frac{3a}{(3au)^2}\right)du$

Since, $\frac{d}{du}\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)=\frac{(3/\sqrt{5})au(-2/\sqrt{5})a-\left((\sqrt{5}/3)-(2/\sqrt{5})au\right)(3/\sqrt{5})a}{(9/5)a^2u^2}$

$=-6/5\,a^2u-a+6/5\,a^2u=-a \Rightarrow$

$I=-\frac{3\sqrt{5}}{3}\int \frac{1}{\sqrt{1-\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)^2}}\cdot\left(-\frac{a}{(3au)^2}\right)du \Rightarrow$

$I=-\sqrt{5}\int \frac{1}{\sqrt{1-\left(\frac{(\sqrt{5}/3)-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)^2}}\frac{d}{du}\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)du$

$\Rightarrow I=-\sqrt{5}\int \frac{1}{\sqrt{1-t^2}}\frac{dt}{du}\,du=+\sqrt{5}\int \frac{-1}{\sqrt{1-t^2}}\,dt \Rightarrow$

$\sqrt{5}\cos^{-1}t=\sqrt{5}\cos^{-1}\left(\frac{\sqrt{5}/3-(2/\sqrt{5})au}{(3/\sqrt{5})au}\right)=\sqrt{5}\cos^{-1}\left(\frac{5/3-2au}{3au}\right)$

$\Rightarrow I=\sqrt{5}\cos^{-1}\left(\frac{5-6au}{9au}\right)=\sqrt{5}\cos^{-1}\left(\frac{5-6(t-2/3)}{9(t-2/3)}\right)=$

$\sqrt{5}\cos^{-1}\left(\frac{1-6t}{9t-6}\right)=\sqrt{5}\cos^{-1}\left(\frac{1-6\sec x}{9\sec x-6}\right)=\varphi$

$\Rightarrow \cos(\varphi/\sqrt{5})=\frac{1-6\sec x}{9\sec x-6}\times\frac{\cos x}{\cos x}\Rightarrow$

$\cos(\varphi/\sqrt{5})=\frac{\cos x-6}{9-6\cos x}=\frac{(-1/3)\cos x-2}{3-2\cos x}$.

(16) B

Find: $\int \frac{dx}{(1+\sin x-\cos x)} = \int \frac{\sec x\,dx}{(\sec x+\tan x)-1}$ | $\frac{d}{dx}(\sec x+\tan x)^{-1} = -(\sec x+\tan x)^{-2}$

$\sec x\tan x + \sec^2 x = \sec x(\tan x+\sec x)$

$$\int \frac{\sec x(\sec x+\tan x)^{-1}}{1-1/(\sec x+\tan x)}\,dx = \int \frac{\sec x(\sec x+\tan x)/(\sec x+\tan x)^2\,dx}{1-1/(\sec x+\tan x)} =$$

let $t = 1/(\sec x+\tan x)$

$$-\int \frac{1}{1-1/(\sec x+\tan x)} \cdot \frac{-d}{dx}(\sec x+\tan x)^{-1}\,dx = -\int \frac{1}{1-t}\,dt = -\ln|1-t|$$

$$= \ln(1-t)^{-1} = \ln\left(1-\frac{1}{(\sec x+\tan x)}\right)^{-1} =$$

$$\ln\left[\frac{\sec x+\tan x-1}{\sec x+\tan x}\right]^{-1} = \ln\left[\frac{\sec x+\tan x}{\sec x+\tan x-1}\right] = \ln\left[\frac{1+\sin x}{1+\sin x-\cos x}\right].$$

Solve: $\int \frac{dx}{\sin x-\cos x-1} = \int \frac{\operatorname{cosec} x\,dx}{1-\operatorname{cosec} x-\cot x} = \int \frac{\operatorname{cosec} x\,dx}{1-(\operatorname{cosec} x+\cot x)} =$

$$I = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x+\cot x)^{-1}\,dx}{(\operatorname{cosec} x+\cot x)^{-1}-1}$$; since, $\frac{d}{dx}(\operatorname{cosec} x+\cot x)^{-1} =$

$(\operatorname{cosec} x+\cot x)^{-2}(-\operatorname{cosec} x\cdot\cot x - \operatorname{cosec}^2 x) =$

$-\operatorname{cosec} x(\cot x+\operatorname{cosec} x) \Rightarrow I = \int \frac{1}{(\operatorname{cosec} x+\cot x)^{-1}-1} \cdot \frac{d}{dx}(\operatorname{cosec} x+\cot x)^{-1}\,dx$

$$= \int \frac{1}{t-1}\,dt = \ln(t-1) = \ln\left[\frac{1-(\operatorname{cosec} x+\cot x)}{\operatorname{cosec} x+\cot x}\right]$$

$$I = \int \frac{dx}{x-2\sqrt{x+2}}$$; let $x = t+2 \Rightarrow dt = dx \Rightarrow I = \int \frac{dt}{t\sqrt{t+4}} = \int \frac{dt}{t^{1/2}\sqrt{t^{1/4}+1}}$

let $t = 4\tan^2 u \Rightarrow dt = 8\tan u\cdot\sec^2 u \Rightarrow$

$$I = \int \frac{8\tan u\cdot\sec^2 u\,du}{2\tan^2 u\cdot\sec u} = 4\int \frac{\sec u}{\tan u}\,du = 4\int \frac{du}{\sin u} = 4\int \sec u\,du =$$

$$-4\tanh^{-1}(\cos u) =$$

$1+\tan^2 u = 1/\cos^2 u \Rightarrow \cos u = 1/\sqrt{1+\tan^2 u} \Rightarrow I = -4\tanh^{-1}(2/\sqrt{4+4\tan^2 u}) =$

$-4\tanh^{-1}(2/\sqrt{4+t}) = I$.

$$\int \frac{dx}{x\sqrt{x^2+x+2}} = \int \frac{\mathbf{1}x}{\sqrt{x^2+x+2}} \cdot \frac{\mathbf{C}}{(cx)^2}\,dx \Rightarrow \frac{x^2+x+2}{(cx)^2} = \left(\frac{ax+b}{cx}\right)^2 + 1 = \frac{(a^2+c^2)x^2+2abx+b^2}{(cx)^2}$$

$a^2+c^2 = 1$, $2ab = 1$, $b^2 = 2 \Rightarrow b = \sqrt{2}$

$2\sqrt{2} = 1/a \Rightarrow a = 1/2\sqrt{2}$; $c^2 = 1-a^2 = 1-\frac{1}{8} = \frac{8}{8}-\frac{1}{8} = \frac{7}{8} \Rightarrow$

$$\int \frac{1\,x}{\sqrt{\left[\frac{1/2\sqrt{2}}{\sqrt{7}/2\sqrt{2}}x+\sqrt{2}\right]^2+1}} \cdot \frac{d}{dx}\left[\frac{x+4}{\sqrt{7}x}\right]dx$$

$x^2+8x+16+7x^2 =$

$8x^2+8x+16 =$

since $\frac{d}{dx}\left(\frac{x+4}{\sqrt{7}x}\right) = \frac{\sqrt{7}x-(x+4)\sqrt{7}}{7x^2} = \frac{\sqrt{7}x-\sqrt{7}x-4\sqrt{7}}{7x^2}$

$$\frac{1}{4\sqrt{7}}\int \frac{\sqrt{8/7}\,x}{\sqrt{\left(\frac{x+4}{\sqrt{7}x}\right)^2+1}} \cdot \frac{d}{dx}\left(\frac{x+4}{\sqrt{7}x}\right)dx = \frac{\sqrt{7}}{4}\cdot\sqrt{8/7}\int \frac{1}{\sqrt{t^2+1}}\,dt = \frac{1}{\sqrt{2}}\sinh^{-1}t =$$

$$I = \frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+4}{\sqrt{7}x}\right).$$

(1) a. If $y = (\sinh^{-1} x)^2$. Show that $(1+x^2)y'' + xy' = 2$

b. Solve $\frac{dy}{dx} + y \cot x = \cos x$

c. Show that $\int_0^{\pi/2} \sin^{2p-1}\theta \, d\theta = \frac{1}{2} B(p,q)$, hence find $\int_0^{\pi/2} \sec^{1/3} x \, dx$

(2) Show that the Gamma function Γ satisfies

$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, $2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)$.

Show also that, $2^{7/5}\pi\left[\Gamma(1/10)\right]^2 = (5+\sqrt{5})\left[\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\right]^2$.

(3) Solve

a. $(D^2+4)y = \sec 2x$, b. $(x^3D^3 - x^2D^2 + 2xD - 2)y = x(x^2+3)$.

(4) a. Put in the normal form, hence solve

a. $y'' - \frac{2}{x}y' + (1+\frac{2}{x^2})y = xe^{2x}$

b. Use the method of variation of parameters to solve

$(D^2+1)y = \operatorname{Cosec} x$.

(5) a. Solve the simultaneous differential equations

$D^2y + z = 99\cos 7x$, $(D^2+5)z - 4y = -36\cos 7x$

b. Explain what is meant by the Wronskian $\Delta(y_1, y_2)$ of the two solutions y_1, y_2 of the differential eqtn $y'' + y'f(x) + yg(x) = 0$ and show that the two solutions are dependent iff $\Delta = 0$.

Show that $\Delta(J_n(x), J_{-n}(x)) = -2\sin(n\pi)/(\pi x)$.

(6) Answer A or B

A. Use $J_n(x) = \sum_{r=0}^{\infty} \frac{(-)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

to show that (i) $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$,

(ii) $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$, (iii) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x)$, (v) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

OR B. Use the relation $(1-2hx+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$

and show that

(i) $P_n(1) = 1$, $P_n(-1) = (-)^n$, $P_{2n}(0) = (-)^n (2n)!/2^{2n}(n!)^2$

(ii) $xP_x' - P_{n-1}' = nP_n$; $P_n' - xP_{n-1}' = nP_{n-1}$; $P_{n+1}' - P_{n-1}' = (2n+1)P_n$.

(18)
B

(1) Solve the following differential equations:

$\frac{dy}{dx} + y \tan x = 4 \cos x$, $(D^2+4)y = \tan 2x$, $(D^2-1)^2 y = x e^x \sin x$

(2) a. Define the Beta function and show that

$\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta = \frac{1}{2} B(p,q)$, Hence find $\int_0^{\pi/2} \sqrt[3]{\tan x}\, dx$.

b. Show that $2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)$, Hence find $\Gamma(\frac{1}{2})$.

c. Show that $\Gamma(x).\Gamma(1-x) = \pi / \sin(\pi x)$.

(3) a. Given $x = e^t$, $D = \frac{d}{dx}$, $\theta = \frac{d}{dt}$, express $x^n D^n$ in terms of θ. Hence solve $(x^3D^3 - x^2D^2 + 2xD - 2)y = x^2$.

b. Solve the differential equation $(D^2+4)y = \sec(2x)$, Using the method of variation of Parameters.

(4) a. Define the Besset function $J_n(x)$ of order n and show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

b. Given that y_1 and y_2 are two solutions of the differential equation $y'' + y' f(x) + y\, g(x) = 0$

Show that the two solutions are independent if their Wronskian $\Delta(y_1, y_2)$ vanishes ($\Delta = 0$). Hence discuss the independence of the two solutions $J_n(x)$, $J_{-n}(x)$ of the Bessel D.E.

(5) Solve the following simultaneous D.E's.

$D^2 x + y = 99 \sin 7t$, $(D^2+5)y - 4x = -36 \sin 7t$.

(6) a. For the Bessel fn $J_n(x)$, Show that

$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$, $2\frac{n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

b. Define the Legendre Polynomial $P_n(x)$ and show that $P_n(-1) = (-1)^n$.

State and prove Rodrigues formulae for $P_n(x)$ and show that

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

2. 1991 "Answer any five questions"

(1) (i) Solve (i) $\frac{dy}{dx} + y\cot x = 2\sin x$, (ii) $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$,
(iii) $(D^2+a^2)y = \sec ax$, (iv) $(D^2+1)y = xe^{-x}\cos x$.

(2) (i) If $D = \frac{d}{dx}$, $\theta = \frac{d}{dt}$ where $x = e^t$, express x^nD^n interms of θ. Hence Solve $[(x+2)^4D^4 + 3(x+2)^3D^3 + 5(x+2)^2D^2]y = 1$.

(ii) Solve the Simultaneous D.E^s

$(D-1)x + 2(D+1)y = \cos t$; $2(D+2)x + (D-3)y = 2\sin t$.

(3) (i) Use the indicated method to Solve

$(D^2+1)y = \cot x$, (Variation of Parameters).

$y'' - 4xy' + (4x^2-1)y = e^{x^2}\cos 2x$ (Reduction to Normal Form).

(ii) Use a suitable definition for $\Gamma(x)$ and Show that

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

(iii) Define the Wronskian $\Delta(y_1, y_2)$ of the 2 solutions y_1, y_2 of the equation $y'' + y'f(x) + y\,g(x) = 0$. Show also that

$$\Delta(J_n(x), J_{-n}(x)) = 2\sin(n\pi)/(\pi x).$$

(4) Show that (i) $nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x).$$

(ii) By using the integral $\int_0^\pi \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ $(a>b)$, Show that

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x + \sqrt{x^2-1}\cos\theta)^n\, d\theta.$$

Hence deduce that, $P_n(-1) = (-1)^n$; $P_2(x) = \frac{1}{2}(3x^2-1)$

(iii) State Rodrigue's formula for $P_n(x)$ and show that

$$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$$

(5) Define $J_n(x)$ as an infinite series and show that

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x), \quad \frac{d}{dx}(x^{-n}J_{-n}(x)) = -x^{-n}J_{n+1}(x).$$

Show also that $\sqrt{\pi x}\,J_{1/2}(x) = \sqrt{2}\sin x$, $\sqrt{\pi x}\,J_{-1/2}(x) = \sqrt{2}\cos x$, and express $J_{3/2}$ interms of x, $\sin x$, $\cos x$.

Show that $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\, d\theta$.

1. 1931 (Solve all)

(20) B

Lm. Dikh

* $b_1 + b_2 = \tanh^{-1}\frac{a_1+a_2}{1+a_1a_2} = \tanh^{-1}a_1 + \tanh^{-1}a_2$ ↦ (when $a_1 = a_2$)

* $\tanh^{-1}a_1 - \tanh^{-1}a_2 = \tanh^{-1}\frac{a_1-a_2}{1-a_1a_2}$

* Solve $y = \left(\frac{2+x}{3-x^2}\right)^{x\sin x}$

$y = \sin^5 4x \cdot \cos^2 7x$, $y = (\sin(mx))^n$

* $y = \left(\frac{2+x}{3-x^2}\right)^{x^2}$, $y = x^{x^x}$, $y = x^{x^{x^x}}$

* prove $\sinh^{-1}x = \ln(x + \sqrt{1+x^2})$

$\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

Generalization

1. $\int \sinh(x+b)\,dx = \cosh(x+b)$

↦ ① the integral does not change if x is replaced by $x+b$

② or $x \mapsto ax$, ③ or $x \mapsto ax+b$

N.Imp Solve $(D^2+4)y = \cos 2x$ by Partial fractions

$y_{P.I} = \left(x + \frac{\sin 4x}{4}\right)\sin\frac{2x}{4} + \frac{1}{16}\cos 4x \cdot \cos 2x$

Notice $I_1 = \frac{1}{D-2i}\cos 2x$, $I_2 = \frac{1}{D+2i}\cos 2x$

for solution ends let $\boxed{(I_1 = I_2)}$

* Solve $(D^2+1)y = 16x\cos x$

$(D^2+4)y = 64x\sin 2x + 32\cos 2x$

$(D^2+4)y = \sec^3 2x$

$(D^2+9)y = 9\cot 3x$

*

$(D^2-3D-4)y = x^2+5x$ by partial fractions and inverse of operator $\Rightarrow (\ )^{-1} =$ taylor expansion

$y_{P.I} =$

Euler

* ① Solve $(x^4D^4 + 3x^3D^3 - 5x^2D^2)y = \cos(2\ln x) + 3$

②

$[(2x+1)^4D^4 + 6(2x+1)^3D^3 + 20(2x+1)^2D^2)y = 16x^2$

* $(x^2D^2 - 4xD + 6)y = x^5$
* $(x^3D^3 - x^2D^2 + 2xD - 2)y = x^3 + 3x$

* ③ $((x+1)^4D^4 + 3(x+1)^3D^3 + 5(x+1)^2D^2)y =$

$3 + \cos[2\ln(x+1)]$

$\Rightarrow y_{G.S} = y_{C.F} + y_{P.I} = y_{C.F} + I + I'$

(1)

(1)

George F. McCrum
Phy. dept.

(Look)

Solutions in pencile written after the exam's 2 hours are over.

11.a| $y = (1+x)^{2x} \Rightarrow \ln y = 2x \ln(1+x) \Rightarrow \frac{y'}{y} = \frac{2x}{1+x} + 2\ln(1+x) \Rightarrow$

$y' = y\left(\frac{2x}{1+x} + 2\ln(1+x)\right) \Rightarrow y' = 2(1+x)^{2x}\left[\frac{x}{1+x} + \ln(1+x)\right]$.

11.b| $y = (\sin^{-1}x)^2 \Rightarrow \sin\sqrt{y} = x \Rightarrow \frac{d}{dx}(\sin\sqrt{y}) = 1 = \cos\sqrt{y} \cdot \frac{1}{2} y^{-1/2} \cdot y'$

$\Rightarrow \cos\sqrt{y}\, y' = 2y^{1/2} \Rightarrow -\cos\sqrt{y} \cdot y' = 2y^{1/2} \Rightarrow$

$-\sin\sqrt{y} \cdot \frac{1}{2} y^{-1/2} \cdot y' \cdot y' + \cos\sqrt{y} \cdot y'' = y^{-1/2} \cdot y' \Rightarrow$

$y'' \cdot \cos\sqrt{y} = y' \cdot y^{-1/2}\left[1 + \frac{1}{2} y' \sin\sqrt{y}\right]$

$2 \times 2y = \cos^2\sqrt{y} \cdot y'^2 = (1 - \sin^2\sqrt{y}) \cdot y'^2 = (1-x^2) y'^2 \Rightarrow$

$2 \times 2y' = (1-x^2) \cdot 2y' y'' + y'^2(-2x) \Rightarrow$

~~$(2y')(2) = 2y'($~~ $y''(1-x^2) = x y') \Rightarrow$

$2 = y''(1-x^2) - x y'$ □

a further look was enough to notice the start of the mistake.

1.c| 1st method using recurrance relations: " $I_n = \frac{n-1}{n} I_{n-2}$ "

$\Rightarrow 2^4 \int_0^{\pi/2} \sin^4\theta \cdot \cos^6\theta \, d\theta = 2^4 \int_0^{\pi/2} (1-\cos^2\theta)^2 \cdot \cos^6\theta \, d\theta =$

$2^4\left[\int \cos^6\theta \, d\theta - 2\int \cos^8\theta \, d\theta + \int \cos^{10}\theta \, d\theta\right] =$

$2^4\left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 2\,\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] =$

$2^4 \cdot \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left[1 - \frac{7}{4} + \frac{9}{10}\cdot\frac{7}{8}\right]$.

2nd Method using Beta function. $I \int_0^{\pi/2} \sin^{2p-1}\theta \cdot \cos^{2q-1}\theta \, d\theta =$

$\frac{1}{2}[\Gamma(p)\cdot\Gamma(q) / \Gamma(p+q)] \Rightarrow 2p-1 = 4 \Rightarrow p = 5/2 \; ; \; 2q-1 = 6 \Rightarrow q = 7/2$

$\Rightarrow \int_0^{\pi/2} \sin^4\theta \cdot \cos^6\theta \, d\theta = \frac{1}{2}[\Gamma(5/2)\cdot\Gamma(7/2) / \Gamma(6)]$

~~$\Gamma(1/2) \times \Gamma(5/2 \times 11) =$~~ $[\Gamma(5/2+1) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \Gamma(7/2)]$

$\Gamma(3/2+1) = \Gamma(5/2) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$; $\Gamma(6) = 5! \Rightarrow$

$\int_0^{\pi/2} \sin^4\theta \cdot \cos^6\theta \, d\theta = \frac{1}{2}\left[\frac{9\cdot5\cdot\pi \times 2^{-5}}{5\times4\times3\times2\times1}\right] = 3\pi \times 2^{-9}$.

$2^{-9}3\pi$

(2)

1.d) $\frac{dy}{dx} + \cot x \cdot y = \operatorname{cosec}^3 x \Rightarrow$

$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \ln(\sin x)$,

$\Rightarrow e^{\int \cot x\, dx} = \sin x \Rightarrow$

$y \cdot \sin x = \int \sin x \cdot \operatorname{cosec}^3 x\, dx = \int \sin^2 x\, dx =$

$\cos 2x = 1 - 2\sin^2 x \Rightarrow$

$\int \frac{1}{2}(1 - \cos 2x)\, dx = \int \sin^2 x\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] \Rightarrow$

$y = \left[\frac{1}{2}\left(x - \left(\sin 2x / 2\right)\right) / \sin x\right]$.

2.(i) $(D^2+1)(D^2+4)y = (D+i)(D-i)(D+2i)(D-2i)y =$

$e^x + e^{-x} \Rightarrow$ Auxiliary eqtn is

$(\lambda+i)(\lambda-i)(\lambda+2i)(\lambda-2i) = 0 \Rightarrow$

$y_{C.F} = A\cos x + B\sin x + A'\cos 2x + B'\sin 2x$

$y_{P.I} = \frac{1}{(D^2+1)(D^2+4)} \underset{I_1}{e^x} + \underset{I_2}{e^{-x}} =$

$\frac{(1+i)(1-i)}{(1+1)\,2} \quad \frac{}{(-1+i)(-1-i)\,2\,1+1\,2}$

$I_1 = \frac{1}{D^2+1} \cdot \frac{e^x}{5} = \frac{1}{5}\cdot\frac{1}{2}e^x = \frac{1}{10}e^x$.

$I_2 = \frac{1}{(D^2+1)(D^2+4)} e^{-x} = \frac{1}{(D^2+1)} \cdot \frac{e^{-x}}{5} = e^{-x}\frac{1}{5\times 2} \Rightarrow$

$y_{G.S} = y_{C.F} + y_{P.I} = (A+A')(\cos x + \cos 2x) + (B+B')(\sin x + \sin 2x) +$

$\frac{1}{10}(e^x + e^{-x}) = y_{G.S}$.

2.(ii) $(D^4 + 18D^2 + 81)y = 0 \Rightarrow$

$(D^2+9)^2 y = 0 \Rightarrow$

$(D^2+9)^2 = 0 \Rightarrow (D+3i)^2(D-3i)^2 = 0 \Rightarrow$

$\lambda = -3i, -3i\ ;\ \lambda = 3i, 3i \Rightarrow$

$y_{G.S} = (Ax+B)e^{3ix} + (A'x+B')e^{-3ix}$

$(Ax+B)(A'\cos 3x + B'\sin 3x) = y_{G.S}$.

(3)

2.(iii) $(D^2+9)y = \sec 3x \Rightarrow (D+3i)(D-3i)y = \sec 3x \Rightarrow$

A.Eqtn is $(\lambda+3i)(\lambda-3i)=0 \Rightarrow$

$y_{C.F} = A\cos 3x + B\sin 3x \Rightarrow$

$y_{P.I} = \frac{1}{D^2+9}\sec 3x = \frac{1}{(D+3i)}\cdot\frac{1}{D-3i}e^{+ix}\cdot(e^{-ix}\sec x)$

$\frac{1}{D^2+9}e^{ix}(e^{-ix}\sec(3x))$

$\frac{1}{6i}\left(\frac{1}{D-3i}-\frac{1}{D+3i}\right)e^{i3x}\,e^{-i3x}\sec 3x = \frac{1}{6i}e^{i3x}\frac{1}{D+3i-3i}e^{-i3x}\sec 3x -$

$\frac{1}{6i}e^{i3x}\frac{1}{D-3i+3i}e^{i3x}\sec(3x) = \frac{1}{6i}e^{i3x}\int(\cos 3x - i\sin 3x)\sec 3x\,dx -$

$-\frac{1}{6i}e^{i3x}\int(\cos 3x + i\sin 3x)\sec 3x\,dx = \frac{e^{i3x}}{6i}\left(x - i\int\frac{\sin 3x}{\cos 3x}dx\right)$

$-\frac{e^{i3x}}{6i}\left(x + i\int\frac{\sin 3x}{\cos 3x}dx\right) = \frac{e^{i3x}}{6i}\left(x+\frac{i}{3}\ln(\cos 3x)\right)$

$\frac{e^{i3x}}{6i}\left(-x+\frac{i}{3}\ln(\cos 3x)\right) = y_{P.I} \Rightarrow$ $\boxed{y_{G.S} = y_{C.F} + y_{P.I}}$

2.(vii) $(D+1)^2 y = x e^{-x} + x^2 \Rightarrow$

A.Eqtn $(\lambda+1)^2 = 0 \Rightarrow \lambda = -1, -1 \Rightarrow$

$y_{C.F} = (Ax+B)e^{-x}$

$y_{P.I} = \frac{1}{(D+1)^2}(e^{-x}x + x^2)$

$I_1 \qquad I_2$

$I_1 = \frac{1}{(D+1)^2}e^{-x}x = \frac{1}{(D+1)^2}e^{-x}(xe^{3x}) = e^{-x}\frac{1}{(D-1+1)^2}xe^{3x} =$

$e^{-x}\iint x e^{3x}(dx)^2 \Rightarrow \frac{1}{3}\int x\frac{3e^{3x}}{3}dx = \frac{1}{3}\int x\,de^{3x} =$

$\frac{1}{3}\left[xe^{3x} - \int e^{3x}dx\right] = \frac{1}{3}\left[xe^{3x} - \frac{1}{3}e^{3x}\right] \Rightarrow$

$\int\frac{1}{3}xe^{3x}dx - \frac{1}{9}\int e^{3x}dx \Rightarrow$

$\left[\frac{1}{3}\cdot\frac{1}{3}\left[xe^{3x} - \frac{1}{3}e^{3x}\right] - \frac{1}{27}e^{3x}\right]e^{-x} = I_1$

$\boxed{e^{2x}\left(\frac{x}{9} - \frac{2}{27}\right) = I_1}$

$I_2 = \frac{1}{(D+1)^2}e^{-x}(x^2e^{x}) = e^{-x}\frac{1}{(D-1+1)^2}x^2e^{x} =$

$e^{-x}\int\left[\int x^2e^{x}dx\right]dx \Rightarrow$

$\int x^2e^{x}dx = x^2e^{x} - 2\int xe^{x}dx = x^2e^{x} - 2(xe^{x} - e^{x}) =$

$x^2e^{x} - 2xe^{x} + 2e^{x}) = \Rightarrow$

$e^{x}\cdot e^{-x}(x^2 - 2x + 2) \Rightarrow$

$y_{G.S} = y_{C.F} + y_{P.I} = y_{C.F} + I_1 + I_2$

$\int x^2e^{x}dx - 2\int xe^{x}dx + 2\int e^{x}dx = e^{x}(x^2-2x+2) - 2e^{x}(x-1) + 2e^{x}$

$\Rightarrow \boxed{I_2} = e^{-x}\cdot e^{x}(x^2-2x+2-2x+2+2) = x^2 - 4x + 6$

(4)

3.(i) $\Gamma(x) = \lim_{n\to\infty} \left[(n-1)!\, n^x \Big/ \prod_{k=0}^{n-1} (x+k) \right]$ =

$$\frac{\Gamma(x+1)}{\Gamma(x)} = \frac{(n-1)!\, n^x \cdot n}{(n-1)!\, n^x} \cdot \prod_{k=0}^{n-1} \left(\frac{x+1+k}{x+k} \right) =$$

$$\prod_{k=0}^{n-1} \frac{x+k+1}{x+k} = \prod_{k=0}^{n-1} \left(1 + \frac{1}{x+k} \right)$$

$$\Gamma(x+1) / \Gamma(x) = \frac{(n-1)!\, n^x \cdot n}{(n-1)!\, n^x} \cdot \prod_{k=0}^{n-1} \left(\frac{x+k}{x+k+1} \right)$$

$$\frac{\prod_{k=0}^{n-1} (x+k)}{\prod_{k=0}^{n-1} (x+k+1)} = \frac{x\,(x+1)(x+2)\cdots(x+n-1)}{(x+1)(x+2)\cdots(x+n-1)(x+n)} =$$

$$\frac{x}{x+n} \Rightarrow \Gamma(x+1) \cdot \Gamma(x) = \lim_{n\to\infty} x \cdot \frac{n}{x+n}$$

$$\lim_{n\to\infty} \frac{n}{n+x} = 1 \Rightarrow \Gamma(x+1) = x\,\Gamma(x) \qquad \square$$

[Written after I gave a look to that part in the lecture' notes.]

3.i Legendre's Duplication formula

$$1/\Gamma(x) = x\, e^{\gamma x} \left(\prod_{k=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \right) = \text{Weirstrass Def. of } \Gamma(x)$$

Legendre Duplication formula $\Rightarrow$

$$m^{mx} \prod_{k=0}^{m-1} \Gamma\left(x + \frac{r}{m}\right) / \sqrt{m} \cdot \sqrt{(2\pi)^{m-1}} \quad \Gamma(mx)$$

when $m=2$ $\Rightarrow$

$$2^{2x}\, \Gamma(x) \cdot \Gamma\left(x+\tfrac{1}{2}\right) = \left(\sqrt{2}\right)^2 \sqrt{\pi}\ \Gamma(2x) \Rightarrow$$

$$2^{2x-1}\, \Gamma(x) \cdot \Gamma\left(x+\tfrac{1}{2}\right) = \sqrt{\pi}\ \Gamma(2x)$$

Let $\varphi(x) = m^{mx} \dfrac{\prod_{r=0}^{m-1} \Gamma\left(x + \frac{r}{m}\right)}{\Gamma(mx)}$:

$$\Gamma(mx) = \lim_{n\to\infty} \left[(mn-1)!\, (mn)^{mx} \Big/ \prod_{k=0}^{mn-1} (mx+k) \right]$$

$$\Gamma\left(x + \frac{r}{m}\right) = (n-1)!\, n^{(x+r/m)} \Big/ \prod_{k=0}^{n-1} (mx + r + k)$$

Substituting in $\varphi(x)$ when $m=2$ we find in the end

$$\varphi(x) = \varphi(1/2) = \varphi(1/m)$$

$$\varphi(x/m) = m\, \Gamma(1/m) \cdot \Gamma(2/m) \cdots \Gamma\left(\frac{m-1}{m}\right)$$

$$\varphi(1/m) = m\, \Gamma(1-1/m) \cdots \Gamma(2/m) \Gamma(1/m)$$

Since $\Gamma(x) \cdot \Gamma(1-x) = \pi / \sin(\pi x) \Rightarrow$

$$\varphi(1/m) = m \cdot \sqrt{\pi}^{m-1} \cdot \frac{\sqrt{2}^{m-1}}{\sqrt{m}} = \sqrt{m} \cdot \sqrt{(2\pi)^{m-1}}$$

$$\prod_{r=1}^{m-1} \sin(r\pi/m) = \frac{m}{2^{m-1}}$$

when $m=2$ $\Rightarrow$ Legendre Duplication formula.

(5)

3.(ii) 3.(iii) $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \Rightarrow$

to solve $\int_0^\infty e^{-x^2}\, dx \Rightarrow$

Let $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{1}{2} t^{-1/2}$

$\Rightarrow \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2}\, dt = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\cdot\sqrt{\pi}$

Back to question 3.(i) 2nd part

Prove that $\Gamma(x+1) = x\,\Gamma(x)$ using $\int_0^\infty e^{-t} t^{x-1}\, dt = \Gamma(x)$

$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x+1-1}\, dt = \int_0^\infty e^{-t} t^{x-1} \cdot (t)\, dt$

Required to Prove: $\int_0^\infty e^{-t} t^{x+1-1}\, dt = \int_0^\infty e^{-t} t^x\, dt = x\int_0^\infty e^{-t} t^{x-1}\, dt$

Since x is independent of the integration operation at the R.H.S

$\Rightarrow$ we can deal with x as a constant $\Rightarrow$

$\int_0^\infty e^{-t} \cdot x\, t^{x-1}\, dt$, but $\frac{d}{dt}(t^x) = x\, t^{x-1}$

$\Rightarrow dt^x = x\, t^{x-1}\, dt$. ($\int u\, dv = uv - \int v\, du$)

$\Rightarrow \int_0^\infty e^{-t}\, dt^x = e^{-t} t^x \Big|_0^\infty - \int_0^\infty -e^{-t} t^x\, dt \Rightarrow$

Since e^{-t} decreases to zero much more than $t^x \Rightarrow$ as $t \to \infty$

$e^{-t} \to 0 \Rightarrow e^{-t} t^x \to 0$

$0 \times 1 = 0 \Rightarrow e^{-t} t^x \Big|_0^\infty = 0$

$\Rightarrow x\int_0^\infty e^{-t} t^{x-1}\, dt = x\,\Gamma(x) = \int_0^\infty e^{-t} t^x\, dt = \int_0^\infty e^{-t} t^{(x+1)-1}\, dt$

$= \Gamma(x+1)$

□

(6)

3.(iii) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta \, d\theta \Rightarrow$

$2p-1 = 1/2 \Rightarrow 2p = 3/2 \Rightarrow p = 3/4$; $2q-1 = -1/2 \Rightarrow$

$q = 1/4 \Rightarrow$

$$= \frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1/4+3/4)} = \frac{1}{2}\frac{\Gamma(3/4)\cdot\Gamma(1-3/4)}{1} = \frac{1}{2}\left(\pi/\sin(3\pi/4)\right)$$

$\sin\frac{3\pi}{4} = \sin 3\times 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}} \Rightarrow$

$$\frac{1}{2}\cdot\frac{\pi}{1}\times\frac{\sqrt{2}}{1} = \frac{\pi}{\sqrt{2}}$$

3.(iv) Bessel DE $x^2y'' + xy + (x^2 - n^2)y = 0$

$$J_n(x) = \sum_{r=0}^{\infty}\left[(-)^r (x/2)^{n+2r} / r!\,\Gamma(n+r+1)\right]$$

$\Rightarrow$

$$x^{-n} J_n x = 2^{-n}\sum_{r=0}^{\infty}\left[(-)^r (x/2)^{2r} / r!\,\Gamma(n+r+1)\right]$$

$$\Rightarrow D(x^{-n} J_n(x)) = 2^{-n}\sum_{r=1}^{\infty}(-)^r \frac{2r\,(x/2)^{2r-1}\times 1/2}{r!\;\Gamma(n+r+1)}$$

Let $r-1 = s \Rightarrow r = s+1 \Rightarrow ((n+1)+s+1) = n+r+1$

$2s+2-1 = 2s+1$

$$\frac{d}{dx}(x^{-n} J_n(x)) = x^{-n}\times(-)\times\sum_{s=0}^{\infty}\frac{(-)^s (x/2)^{(n+1)+2s}}{s!\,\Gamma((n+1)+s+1)} =$$

$$= x^{-n} J_{n+1}(x) \quad \square$$

Solution in blue pen are done without any aid of the lectures ie direct from memory.

Gamma Function

1. Integral Definition

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt$$

2. Euler Definition

$$\Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{\prod_{k=0}^{n-1}(x+k)}$$

3. Weierstrass Definition

$$\frac{1}{\Gamma(x)} = x e^{\gamma x}\left(\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right)$$

4. $\Gamma(n+1) = n!\,, \quad n \in \mathbb{Z}^+$

5. $$\Gamma(Px) = \lim_{Pn\to\infty} \frac{(Pn-1)!\,(Pn)^{Px}}{\prod_{k=0}^{Pn-1}(Px+k)}$$

1. Properties of $\Gamma(x)$

1. Difference formula

$$\Gamma(x+1) = x\Gamma(x)$$

2. $$\Gamma(x)\Gamma(1-x) = \pi / \sin(\pi x)$$

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n}\right)^2\right)$$

3. $$\varphi(x) = \frac{m^{(mx)}\left[\prod_{r=0}^{m-1}\Gamma\left(x+\frac{r}{m}\right)\right]}{\Gamma(mx)} = \frac{\sqrt{m}}{\sqrt{(2\pi)^{m-1}}}$$

Beta Function

1. Integral Definition

$$B(P,q) = \int_0^1 t^{P-1}(1-t)^{q-1}\, dt$$

1. Properties of $B(P,q)$

$\int u\,dv = uv - \int v\,du$ (IBP)

1. Symmetric $B(P,q) = B(q,P)$

$(1-t) = T$

2. $P B(P, q+1) = q B(P+1, q)$

($\int_0 dt = d\Omega$, IBP)

3. $B(P, q+1) + B(P+1, q) = B(P, q)$

4. $$B(P,q) = \frac{\Gamma(P)\Gamma(q)}{\Gamma(P+q)}$$

5. when $t = \sin^2\theta$, $1-\sin^2\theta = \cos^2\theta$

$$\rightarrow \frac{1}{2}B(P,q) = \int_0^{\pi/2} \sin^{2P-1}\theta \cos^{2q-1}\theta\, d\theta$$

$$\prod_{k=1}^{m-1} \sin\left(\frac{\pi k}{m}\right) = \frac{m}{2^{m-1}}$$

4. when $m = 2 \rightarrow$

1. $2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x)$

when $x = \frac{1}{10} \rightarrow$ ⑧ $\Gamma\left(\frac{2}{5}\right)$

2. $\sin\theta = \sin 72 = \sin(90-18) = \cos 18$

$\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$

, $\varphi = 18° \rightarrow \sin 2\varphi = \cos 3\varphi$

3. $\sin 72° = \cos 18° = \sqrt{1-\sin^2 18} \rightarrow$

$2\sqrt{\pi}\,\Gamma\left(\frac{1}{10}\right) = \sqrt{5+\sqrt{5}}\,\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)$

Bessel functions

1. Bessel's D.E

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

2. $$J_{\pm n} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\pm n + r + 1)} \left(\frac{x}{2}\right)^{\pm n + 2r}$$

3. General Solution to 1

$$y_{G.S} = AJ_n + BJ_{-n}$$

and $n \notin \mathbb{Z}$

.1 Generating function

$$e^{\frac{x}{2}\left(u - \frac{1}{u}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, u^n$$

and $n \in \mathbb{Z}$

$$e^{\varphi} = \sum_{k=0}^{\infty} \frac{\varphi^k}{k!} \Rightarrow \left(u - \frac{1}{u}\right)^k = \sum_{n=0}^{k} \binom{k}{n} u^{k-n} \left(\frac{1}{u}\right)^n$$

.2 Bessel's Definite Integral, $J_n(x)$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta$$

.3 Condition of Linear Independence

From 1 $\rightarrow$ $y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$

The Wronskian Δ is:

$$\Delta(J_n, J_{-n}) = A\, e^{-\int \frac{1}{x}dx} = A\,\frac{1}{x} =$$

$$J_n' J_{-n} - J_n J_{-n}' = \frac{2}{x}\cdot\frac{1}{\pi}\sin(n\pi)$$

1 $\mapsto$ $A = 0$ when $n \in \mathbb{Z}$ $\rightarrow \Delta = 0$

$\Rightarrow$ Linearly dependent solutions

Legendre's Polynomials

1. Legendre's D.E

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

2. $$P_n = \frac{1}{2^n} \sum_{r=0}^{m} \frac{(-1)^r (2n-2r)!}{r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$$

$$m = \begin{cases} n/2 & \rightarrow n \text{ even} \\ (n-1)/2 & \rightarrow n \text{ odd} \end{cases}$$

3. Rodrigues formula

$$P_n = \frac{1}{2^n n!} D^n (x^2 - 1)^n$$

.1 Generating function

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\, h^n$$

$$\lambda = (2xh - h^2)$$

$$(1 - \lambda)^{-\frac{1}{2}} = 1 + \frac{1}{2}\lambda + \frac{1\cdot 3}{2\cdot 4}\lambda^2 + \dots$$

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}} \qquad (a = 1 - hx;\ b = -h\sqrt{x^2 - 1})$$

.2 Laplace Integral for $P_n(x)$

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{x^2 - 1}\cos\theta\right)^n d\theta$$

.3 Condition of Linear Independence

$$\int_{-1}^{+1} P_m(x) P_n(x) = \begin{cases} 0, & n \neq m \\ \dfrac{2}{2n+1}, & n = m \end{cases}$$

We used the Rodrigues and $D^n(fg) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} D^{n-k} g(x)\, D^k f(x)$

2 When $n \in \mathbb{R} - \mathbb{Z} \rightarrow$

$\Delta \neq 0$, $\Delta = \dfrac{2 \sin n\pi}{\pi x}$

$= \Delta (J_n , J_{-n})$ (we utilized $\S$ - $\S^*$)

$\Rightarrow$ two linearly independent solutions

3 $J_{-n} = (-)^n J_n$, $n \in \mathbb{Z}^+$

1. Recurrence Relations

1 $D(x^{\pm n} J_{\pm n}) = \pm x^{\pm} J_{n\mp 1}(x)$ (there ; here)

2 $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

3 $2 \dfrac{n}{x} J_n = J_{n-1} + J_{n+1}$

$\sin x = \sum_{r=0}^{\infty} \dfrac{(-1)^r x^{2r+1}}{(2r+1)!}$

$\cos x = \sum_{r=0}^{\infty} \dfrac{(-1)^r x^{2r}}{(2r)!}$

1. Further Properties

1 $x^{-(m+n)} J_{m+n} = (-1)^m \left[\dfrac{D}{x}\right]^m (x^{-n} J_n)$

[From directly (4.3) $m+n = \frac{3}{2}+\frac{5}{2}$] or [(4.1) $\rightarrow$ as a Proof (m times)]

$\Rightarrow$ when $n = \pm\frac{1}{2}$ and $m \in \mathbb{Z}^+$

2 $x^{-(m \pm \frac{1}{2})} J_{m \pm \frac{1}{2}} = (-1)^m \sqrt{\dfrac{2}{\pi}} \left(\dfrac{D}{x}\right)^m \left(\dfrac{\sin x}{x}\right)$

$\rightarrow$ (No. of factors of $\Gamma(\ )$)

$J_{\frac{1}{2}} = \sqrt{\dfrac{2}{\pi x}} \sin x$; $J_{-\frac{1}{2}} = \sqrt{\dfrac{2}{\pi x}} \cos x$

(evens) + (odds) (G.F.) [$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $u = e^{i\theta}$]

3 From 0.1 $\rightarrow e^{i(x \sin\theta)} = J_0 + 2 \sum_{n=1}^{\infty} J_{2n} \cos 2n\theta$

$+ 2i \sum_{n=0}^{\infty} J_{2n+1} \sin(2n+1)\theta = \cos(x \sin\theta) + i \sin(x \sin\theta)$ $\rightarrow$ by expansion

Special Cases occur when (1) $\theta = 0$, (2) $\theta = \pi/2$, (3) $\theta = \pi$, (4) , $x = \theta$)

which is Leibnitz's formula for the n^{th} derivative of (gf)

2 The Orthogonal Property of $P_n(x)$ w.r.t. x^k is (From 0.3 and 1.6P) $\rightarrow D^{n-m-1}$

$\int_{x=-1}^{+1} x^k P_n(x)\,dx = 0$, $k = 0, 1, \ldots, n-1$

1. Recurrence Relations

1 $n P_n - x(2n-1) P_{n-1} + (n-1) P_{n-2} = 0$

2 $x P_n' - P_{n-1}' = n P_n$

3 $P_n' - x P_{n-1}' = n P_{n-1}$

4 $P_{n+1}' - P_{n-1}' = (2n+1) P_n$

5 $x \times (3) - (2) \Rightarrow$

$(1 - x^2) P_{n-1}' = nx P_{n-1} - n P_n$

1. Further Properties

1 $P_n(\cos\theta) = \left[\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]$ (from 0.1, G.F.) , $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\left[2 \cos n\theta - \dfrac{1 \cdot (2n)}{2(2n-1)} 2 \cos(n-2)\theta + \left(\dfrac{1 \cdot 3 \cdot (2n) \cdot (2n-2)}{2 \cdot 4 \cdot (2n-1) \cdot (2n-3)}\right) 2 \cos(n-4)\theta + \cdots\right.$

$\left.\pm \frac{1}{2}\right]$ $\rightarrow$ expansion $\rightarrow$ Comparison of n^{th} terms in LHS = RHS

2 factors

1. nth order, Constant Coefficient

$$\sum_{k=0}^{n} a_k D^k y = R(x)$$

1 when $R(x)=0$ → Homogeneous

$n=2 \rightarrow D^2y + PDy + q = 0$

$y_{C.F} = Au + Bv$ ($y = u$, v are L.I.P)

2 when $D^2y + 2aDy + b = 0$

let $y = e^{\lambda x}$

Auxiliary Polynomial is $\lambda^2 + 2a\lambda + b = 0$

1- $\lambda_1 \neq \lambda_2 \rightarrow y_{C.F} = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

2- $\lambda_1 = \lambda_2 \rightarrow y_{C.F} = (A+Bx)e^{\lambda_1 x}$

3- $\lambda_1, \lambda_2 = \lambda_1^* \in \mathbb{C} \rightarrow \lambda = a \pm ib$

$y_{C.F} = e^{ax}(A\cos bx + B\sin bx)$

4- when $n > 2$, λ_i is of multiplicity $k \rightarrow y_{C.F} = \left(\sum_{r=0}^{k} a_r x^r\right) e^{\lambda_i x} + \square$

$y_{C.F}$ = Complementary function

1 When $R(x) \neq 0$

$\rightarrow y_{G.S} = y_{C.F} + y_{P.I}$

To obtain $y_{P.I}$ particular integral

$\rightarrow F(D)y = R(x) \rightarrow$

$y = \frac{1}{F(D)} R(x) = \varphi(D) R(x)$

(1) $\varphi(D)[e^{\alpha x}] = e^{\alpha x}\varphi(\alpha)$

(2) $\varphi(D^2)\begin{bmatrix}\sin\alpha x \\ \cos\alpha x\end{bmatrix} = \begin{matrix}\sin\alpha x\, \varphi(-\alpha^2) \\ \cos\alpha x\, \varphi(-\alpha^2)\end{matrix}$

(3) $\varphi(D)[e^{\alpha x} v] = e^{\alpha x}\varphi(D+\alpha)v$

$D^n \ln(ax+b) = (-)^{n-1}(n-1)!\, a^n / (ax+b)^n$

2 Rules to Solve $\varphi(D)(R(x))$

1 Avoid $\varphi(D)$ in (1), (3) to be ∞

2 Make use of Partial fractions → (1) & (3) by factoring the denominator.

Constant $C = Ce^{-\alpha x}e^{\alpha x}$

4 If the denominator Contains more than on factor → Solve 1st for the factors which does not Cause the appearance of ∞.

5 If the remaining factors may Cause ∞ → use (3) to solve by inserting a Constant.

6 If $R(x) = x^k \rightarrow$ expand some of the denominator factors

$(1-\lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots$

$(1+\lambda)^{-1} = 1 - \lambda + \lambda^2 - \lambda^3 - \cdots$

7 $R[e^{i\theta}] = \cos\theta$; $\mathrm{Im}[e^{i\theta}] = \sin\theta$

→ You may factor the denominator and use $\varphi R[e^{i\alpha x}] = R\varphi[e^{i\alpha x}]$

8 If $n > 2$ and difficult to get the differend make a guess on λ s.t $\lambda \in \{\pm i, \pm 1, \pm 2, \text{small numbers}\}$ → obtain $y_{C.F}$

9 If $R(x)$ Contain more than one factor use (3) by arranging $R(x)$ in Such a way that (3) is applied directly.

10 we Can use (2) with a polynomial

D^k, k odd $\rightarrow D(D^{k-1}((-\alpha^2)))$

11 Multiplication by the Conjugate operator φ^+ and then use (2)

2. Euler's Homogeneous D.E's

$$\left(\sum_{k=0}^{n} a_k x^k D^k\right) y = 0$$

1. Let $x = e^t \Rightarrow D = \frac{dt}{dx}\frac{d}{dt} \Rightarrow$
$xD = \theta = \frac{d}{dt} \rightarrow$

2. $(xD)^k = \prod_{r=1}^{k} (\theta - (r-1))$

e.g. $(xD)^2 = x^2 D \cdot D = x^2 D\left(\frac{x}{x} D\right)$
$= x^2 D\left(\frac{1}{x}\theta\right) \sim = \theta(\theta - 1)$

1. Generalized Euler eqtns

$$\left[\sum_{k=0}^{n} a_k (a+bx)^k D^k\right] y = F(x)$$

1. $(a+bx) = e^t \rightarrow \theta = \frac{d}{dt}$
$((a+bx)D)^2 = b^2 \theta(\theta - 1)$
$((a+bx)D)^k = b^k \left[\prod_{r=1}^{k} (\theta - (r-1))\right]$

3. Simultaneous L.D.E's

with operators of Constant Coefficients →

$$f(D)\,x + \varphi(D)\,y = F(t)$$
$$\psi(D)\,x + \chi(D)\,y = G(t)$$

$$\begin{bmatrix} f(D) & \varphi(D) \\ \psi(D) & \chi(D) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F(t) \\ G(t) \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} F(t) & \varphi(D) \\ G(t) & \chi(D) \end{vmatrix}} = \frac{y}{\begin{vmatrix} f(D) & F(t) \\ \psi(D) & G(t) \end{vmatrix}} = \frac{1}{\begin{vmatrix} f(D) & \varphi(D) \\ \psi(D) & \chi(D) \end{vmatrix}}$$

Variation of Parameters

4. Normal Form

$$D^2 y + P(x) Dy + Q(x) y = R(x)$$

1. let $y = y_1 v$, and y_1 is an already known solution

For $\Rightarrow D^2 y_1 + P D y_1 + Q y_1 = 0$

$\rightarrow y_1 v'' + (2y_1' + P y_1) v' = R \rightarrow$

let $v' = w \rightarrow$

$$y_1 w' + (2y_1' + P y_1) w = R \quad (1)$$

which is a linear eqtn of 1st order in w. (Integrating factor)

2. $2y_1' + P y_1 = 0 \rightarrow y_1 = e^{-\frac{1}{2}\int P\,dx}$

3. (1) is called the Normal form

4. → obtain $Dy = D y_1 v$ and $D^2(y_1 v)$

This is the Reduction of Order Method.

5. Method of Variation of Parameters

Having know $y_{C.F}$ of

$$[D^2 + P(x) D + Q(x)]\, y = R(x)$$

in the general form →

$y_{C.F} = C_1 y_1 + C_2 y_2 \rightarrow \approx$

$C_1 \mapsto u$, $C_2 \mapsto v$

y_1, y_2 are for simplicity $\sin x, \cos x, e^x$ as results from solving for $y_{C.F}$.

1. Having obtained $Dy = y'$ → let $u'y_1 + v'y_2 = 0$ (1)

2. Obtain $D^2y = y''$ → use 1° and the fact that y_1 and y_2 are Complementary functions →

3. Insert in $[D^2 + P(x)D + Q(x)]y = R(x)$ we obtain

4. $$u'y_1' + v'y_2' = R(x) \quad (2)$$

5. From (1) and (2) we seek a solution to obtain u and v

A.S. Univ. Faculty of Science

Third Year (Physics Section) Diff. Eqns, Special Fns. Time: 3 hours

May 92

1. a. Give a suitable definition for $\Gamma(x)$ and show that

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \text{ and } \sqrt{\pi} = \Gamma(1/2).$$

Show also that $2^{2x-1}\,\Gamma(x)\,\Gamma(x+1/2) = \sqrt{\pi}\,\Gamma(2x)$

Find a similar expression for $3^{3x}\,\Gamma(x)\,\Gamma(x+\frac{1}{3})\,\Gamma(x+\frac{2}{3})$

b. Define the Beta function $B(p,q)$ and show that

$$B(p,q+1) + B(p+1,q) = B(p,q), \quad \int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta\, d\theta = \frac{1}{2} B(p,q).$$

2. Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$, $(D^2+a^2)y = \sec ax$, $(D^4-16)y = \sin 2x$

3. If $x = e^t$, $D \equiv \frac{d}{dx}$, $\theta = \frac{d}{dt}$, show $x^n D^n = \theta(\theta-1)\ldots(\theta-n+1)$

Hence solve $\left[(x+5)^3 D^3 - (x+5)^2 D^2 + 2(x+5)D - 2\right] y = (x+5)^4$

4. a. Put in the normal form hence solve $x^2y'' - 2xy' + (x^2+2)y = x^3 e^{2x}$

b. Use the variation of parameter method to solve

$$(D^2+4)y = \cot 2x$$

c. Using the Wronskian of Bessel's differential equation to discuss the independence of the two solutions $J_n(x)$, $J_{-n}(x)$.

5. a. Solve the two simultaneous differential equations

$$D^2x + y = 99\cos 7t, \quad (D^2+5)y - 4x = -36\cos 7t$$

b. Show that

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x),$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

c. Find $J_{1/2}(x)$, $J_{-1/2}(x)$ in terms of $\sin x$, $\cos x$.

6. State and prove Rodrigue's formulae for $P_n(x)$ and show that

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

Show that $P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2-1}\cos\theta)^n\, d\theta$ and find $P_4(x)$.

N.B. Attempt ALL Questions

E.R. Yacoub

① a. Find y' given $y = (1+x)^{2x}$.

b. If $y = (\sin^{-1}x)^2$, show that $(1-x^2)\,y'' - x\,y' = 2$.

c. Use two different methods to evaluate $\int_0^{\pi/2} (\sin 2\theta)^4 \cos^2\theta \, d\theta$

d. Solve $y' + y\cot x = \operatorname{cosec}^3 x$.

② Solve (i) $(D^4 + 5D^2 + 4)y = e^x + e^{-x}$, (ii) $(D^4 + 18D^2 + 81)y = 0$

(iii) $(D^2 + 9)\,y = \sec 3x$, (iv) $(D+1)^2 y = xe^{2x} + x^2$

③ (i) Use Euler's definition to show that $\Gamma(x+1) = x\,\Gamma(x)$. Can you obtain same result by using the relation $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$

(ii) State and prove Legendre's Duplication formulae for $\Gamma(x)$, and shed $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

(iii) By using the Beta function, show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \pi/\sqrt{2}$.

(iv) Define Bessel function and show that $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$.

Ain Shams University — Third year Exam.

Time allowed: 3 hours — Diff. eqn. & Special Functions — Sep. 1992

1) Define $\Gamma(x)$ and show that $2^{2x}\Gamma(x)\Gamma(x+\frac{1}{2}) = 2\sqrt{\pi}\,\Gamma(2x)$ and deduce a similar expression for $3^{3x}\Gamma(x)\Gamma(x+\frac{1}{3})\Gamma(x+\frac{2}{3})$.

Use the definition $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ and show that

$$\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta\, d\theta = \frac{1}{2}B(p,q)$$

and find $\int_0^{\pi/2} \sqrt{\tan x}\, dx$.

2) Define $J_n(x)$ as a power series in x and use this definition to find $J_{1/2}(x)$, $J_{-1/2}(x)$ in terms of x, $\sin x$ and $\cos x$.

Establish two formulae for $\frac{d}{dx}(x^n J_n(x))$, $\frac{d}{dx}(x^{-n} J_n(x))$ and hence or otherwise find $J_{3/2}(x)$, $J_{-3/2}(x)$.

3) Define the Legendre's polynomial $P_n(x)$ and establish Rodrigue's formulae for $P_n(x)$. Show also that

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

Write a definite integral for $P_n(\cos\theta)$ and find $P_2(\cos\theta)$.

4) Solve the following differential equations

$$(D^2+2)y = 2\tan\sqrt{2}x \quad ; \quad (D-4)^3 = x e^{3x} + 1$$

$$x^3D^3 - x^2D^2 - 4xD + 4 = 0$$

5) Use the method of variation of parameters to solve

$$(D^2+9)y = 9\cot 3x$$

Find the general solution of the differential equation

$$x^2y'' - xy' = e^{2x}$$

Show that $y = x^2$ is a solution of the complementary function.

6) (i) Solve the following differential equation $(D^4+16)y = \cos 2x$

(ii) $(D^4-81)y = \cosh 3x + \sinh 3x$.

5 questions

Full Marks may be obtained for about five questions.